V.V. Bolotin

Nonconservative Problems of the Theory of Elastic Stability

В. В. БОЛОТИН

НЕКОНСЕРВАТИВНЫЕ ЗАДАЧИ ТЕОРИИ УПРУГОЙ УСТОЙЧИВОСТИ

V. V. BOLOTIN

Nonconservative Problems of the Theory of Elastic Stability

(CORRECTED AND AUTHORIZED EDITION)

TRANSLATED FROM THE RUSSIAN BY

T. K. LUSHER

ENGLISH TRANSLATION EDITED BY

G. HERRMANN

NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS

PERGAMON PRESS
OXFORD · LONDON · NEW YORK · PARIS
1963

Headington Hill Hall, Oxford
4 and 5 Fitzroy Square, London, W.1

PERGAMON PRESS INC.
122 East 55th Street, New York 22, N.Y.

GAUTHIER-VILLARS ED.
55 Quai des Grands-Augustins, Paris, 6

PERGAMON PRESS G.m.b.H.
Kaiserstrasse 75, Frankfurt am Main

Distributed in the Western Hemisphere by
THE MACMILLAN COMPANY · NEW YORK
pursuant to a special arrangement with
Pergamon Press Limited

Copyright © 1963
PERGAMON PRESS LTD.

Library of Congress Catalog Card Number 63-21133

A translation of the original volume
"NEKONSERVATIVNYIE ZADACHI TEORII UPRUGOI USTOICHIVOSTI",
Fizmatgiz, Moscow (1961) with additions and revision by the author

Made in Great Britain

CONTENTS

PREFACE ix

AUTHOR'S PREFACE TO ENGLISH EDITION xi

TRANSLATION EDITOR'S PREFACE TO ENGLISH EDITION xii

INTRODUCTION

1. Evolution of the problem of elastic stability 1
2. Longitudinal bending of an axially compressed bar 5
3. Bar under the action of a "follower" force. Euler's method 7
4. Stability with respect to small disturbances 9
5. Critical value of "follower" force 11
6. Critical value of "follower" force (continued). System with two degrees of freedom 12
7. Discussion of results. Potential of external forces 15
8. Range of problems covered in the book 18

CHAPTER 1. General Principles

1.1. Introductory remarks 25
1.2. Finite strain 28
1.3. Equilibrium equations and boundary conditions 30
1.4. Geometrical interpretation of results 33
1.5. Relation between stresses and strains 36
1.6. Curvilinear coordinates 36
1.7. Equations of the non-linear theory of elasticity in an arbitrary system of curvilinear coordinates 41
1.8. Formulation of the stability problem. Variational Equations 43
1.9. Various cases of load behavior 47
1.10. Static boundary-value problem 49
1.11. Oscillations about the equilibrium position and Euler's method 55
1.12. Reduction to a system of ordinary differential equations 58
1.13. Evaluation of coefficients for certain particular systems 63
1.14. Investigation of stability 72
1.15. Example. System with two degrees of freedom 75
1.16. Effect of dissipative forces on stability 79

CHAPTER 2. Stability of equilibrium of elastic systems in the presence of follower forces

2.1. Historical background 86
2.2. Problem of the stability of a bar compressed by a tangential force 90
2.3. Influence of mass distribution 93
2.4. Approximate solution of the problem 95
2.5. Effect of damping on stability 98
2.6. Problem of the stability of a bar compressed by a force with a fixed line of action 100
2.7. Stability of the plane form of bending (derivation of the equations) 104
2.8. Some numerical results 111
2.9. Some further problems 115
2.10. Equations of equilibrium of a bar in compression and torsion 119
2.11. Stability of the rectilinear form of a bar in compression and torsion (Euler's method). Classification of the boundary conditions 124
2.12. Bar with a concentrated mass at the end. Method of small oscillations 131
2.13. Effect of the distributed mass of the bar and of damping 134

CHAPTER 3. Stability of flexible shafts with controlled speed of revolution

3.1. Introductory remarks 139
3.2. Equations of motion of a flexible shaft 143
3.3. Viscous internal friction. Instability caused by internal friction 146
3.4. Friction independent of velocity 150
3.5. The case of arbitrary dependence of friction on frequency 155
3.6. Generalization of the problem to the case of unequal principal stiffnesses and an infinite number of degrees of freedom 157
3.7. Non-linear problem 161
3.8. Steady asynchronous precession 166
3.9. Examples of amplitude relations 171
3.10. Friction caused by macroscopic thermal diffusion 176
3.11. Effect of frictional forces in the case of components shrunk on a shaft 184
3.12. Instability of rotors due to the effect of an oil layer in the bearings 186
3.13. Instability in centrifuges incompletely filled with liquid 191
3.14. Instability of rotors in a magnetic field 194

CHAPTER 4. Stability of elastic bodies in a gas flow

4.1. Short historical introduction 199
4.2. Flutter of a wing as a non-conservative problem of elastic stability 203
4.3. General formulation of problems of stability of elastic bodies in potential gas flow 208

4.4. Stability of an elastic cylindrical shell in compressible gas flow 214
4.5. Case of an infinitely long shell. Various types of flow 218
4.6. Determination of critical flutter and divergence velocities 223
4.7. Stability of elastic plates in potential gas flow 231
4.8. Determination of aerodynamic forces in the case of high supersonic velocities. Law of plane sections 236
4.9. Stability of elastic plates at high supersonic velocities 242
4.10. Application of Galerkin's variational method. Effect of damping and of forces in the middle surface 247
4.11. Limits of application of Galerkin's method. Explanation of a paradox in the problem of membrane flutter 257
4.12. Non-linear problems in the theory of aeroelasticity. Effect of geometric and aerodynamic non-linearities 265
4.13. Derivation of the equations of non-linear flutter of a shallow shell at high supersonic velocities. 274
4.14. Approximate method of solution of the equations 280
4.15. Panel supported over its entire contour 285
4.16. Non-linear flutter of a flat panel. Solution by trigonometric series 290
4.17. Small-parameter method for investigation of non-linear flutter 298
4.18. Analysis of results 306

CONCLUDING REMARKS. Suggested directions for future research 313

AUTHOR INDEX 319

SUBJECT INDEX 321

PREFACE

THE present book is devoted to the study of the stability of elastic systems under the action of non-conservative forces. It is wellknown that for such systems the usual methods of the theory of elastic stability, which are based on an examination of forms of equilibrium close to the undisturbed form, are in general no longer applicable. Here we need to use more general methods and more involved means of investigation.

The book contains an introduction and four chapters. The first chapter covers general problems, their formulation and methods of solution. It is based on a paper read by the author at the Third All-Soviet Mathematical Conference in Moscow in 1956. The remaining chapters are devoted to applications. The second chapter considers the stability of elastic systems under the action of non-conservative forces which during the process of loss of stability behave according to some pre-determined law (so called "follower" forces). The third chapter considers the stability of high-speed rotating elastic rotors under the action of various disturbing forces, for example, forces of internal friction, hydrodynamic and electric forces, etc. The fourth chapter deals with problems of stability of elastic systems in a high-speed gas flow; particular attention is paid to the problem of supersonic flutter of elastic plates and shells. A number of problems are considered in non-linear form, which enables the behavior of the system to be studied after loss of stability. It will be seen that all these problems are of considerable interest in present day mechanical, aeronautical and rocket engineering.

A considerable part of the book is devoted to results obtained by the author. Wherever results of other authors have been used the appropriate references are given in a footnote. The author would like to thank E. L. POZNIAK, L. V. EPISHEV, B. P. MAKAROV, Iu. Iu. SHVEIKO, G. V. MISHENKOV and Iu. V. GAVRILOV for their help in carrying out or checking some of the calculations and for providing a number of previously unpublished experimental results. He would also like to express his gratitude to A. I. LUR'E for his very valuable advice on the planning of the book.

PREFACE TO THE ENGLISH EDITION

PRESENTING to the reader an English edition of the book, the author would like to indicate certain corrections and additions. First, the introduction has been revised, and is developed now along somewhat different methodological principles. Secondly, certain formulations have been made more rigorous and all noted misprints in the Russian edition have been corrected. Thirdly, there were added references to publications which appeared after the writing of the book or were published earlier, but remained unknown to the author.

The author would like to express his deep appreciation to the following persons: Dr. George Herrmann, who edited the translation of the book; to Professors L. I. Balabukh, A. S. Vol'mir, G. Iu. Dzhanelidze and V. I. Feodos'ev, who communicated their comments to the author; to M. F. Dimentberg, B. P. Makarov, V. N. Moskalenko, Iu. N. Novichkov and Iu. Iu. Shveiko, who re-read the Russian edition of the book and indicated misprints and inaccuracies.

V. V. BOLOTIN

Moscow

TRANSLATION EDITOR'S PREFACE TO ENGLISH EDITION

As a result of recent technological advances, the importance of stability investigations of elastic systems subjected to non-conservative forces has increased considerably during the last decade. Our knowledge of instability phenomena associated with static "follower" forces, with aerodynamic forces, and with inertia and friction forces acting on such systems has been widened significantly. Due to the scatter of research results in many publications and in many languages, it was hardly possible, however, to assess the level of advance made and to make use of the numerous common features in different technological applications, which, in turn, could facilitate further progress. It is in recognition of this need that it was decided to prepare an English translation of a recent book in Russian by Professor V. V. Bolotin, in which non-conservative problems of the theory of elastic stability are treated for the first time in a unified and comprehensive manner.

It is a pleasure to acknowledge in this undertaking the cooperation of Professor V. V. Bolotin, who took it upon himself to bring the contents up to date, as described in greater detail in his own preface to the English edition; of Mr. J. K. Lusher, London, England, who prepared a very competent translation manuscript; of Mr. R. W. Bungay, graduate student at Northwestern University, who read the edited manuscript and made several valuable comments regarding the translation; and finally, of several editors of Pergamon Press, who made a number of useful suggestions regarding the preparation of the translated edition.

GEORGE HERRMANN

Northwestern University
March 1963

INTRODUCTION

1. EVOLUTION OF THE PROBLEM OF ELASTIC STABILITY

The theory of elastic stability, which originated in the works of Euler, is now a very well-developed branch of applied mechanics employing many effective techniques and possessing a large number of problems already solved, as well as a very large body of literature. One of the major factors which contributed to the rapid accumulation of material in the field of elastic stability was undoubtedly the extremely successful concept of stability and critical force. In the theory of elastic stability it is assumed that for sufficiently small loads the equilibrium of an elastic system is stable, and that it remains so up to the first point of bifurcation of equilibrium forms; thereafter, the initial form of equilibrium becomes unstable. The critical force (or, in general, a parameter of the force group) is then defined as the smallest value of the force at which, in addition to the initial form of equilibrium, there can exist others which are very close to the initial form. This concept is to be found as far back as the works of Euler, who defined the critical force as "the force required to cause the smallest inclination of a column"[1]. This approach, or, as we shall refer to it in the future, the *Euler method*, has enabled us to reduce the problem of the stability of a form of equilibrium to the simpler problem of finding the minimum characteristic value of certain boundary-value problems.

The usefulness of the Euler method in the theory of elastic stability cannot be disputed. It has also been extended to problems of non-elastic stability, expressed in the concept of a "reduced modulus". However, the *Euler method* is not universal; its range of application can be clearly defined and a number of mistakes and misunderstandings have occurred in the past as a result of attempts to apply the theory outside this range. Three factors should be remembered in connection with the Euler method in the theory of stability.

---

[1] EULER, L. *Method of finding Curves Possessing the Properties either of a Maximum or of a Minimum, or the Solution of the Isoperimetric Problem in its Widest Sense* (Metod nakhozhdeniia krivykh linii, obladaiushchikh svoistvami libo maksimuma, libo minimuma, ili reshenie izoperimetricheskoi zadachi vziatoi v samom shirokom smysle), p. 491, Gostekhizdat, 1934.

One of these is associated primarily with the development of the non-linear theory of thin elastic shells. As far back as the thirties it was established that in the case of shells a systematic and very significant divergence existed in the values of critical loads given by the classical theory and by experimental results. It was found that for thin shells initial inaccuracies and non-linear effects assume considerable importance, and that the critical forces corresponding to the points of bifurcation of equilibrium are in fact the "upper" critical forces, which are difficult to realize even under the most ideal experimental conditions.[1]

The second factor is associated with the development of the theory of plastic stability. In 1946 Shanley[2] pointed out that the concept of a "reduced modulus" corresponds only to a particular assumption concerning the behavior of the load, and that in the plastic range the critical load should be determined by a different method from that used in the elastic range. Recent investigations have clearly established the important part played by the behavior of the load during the process of loss of stability and, in general, the part played by the time factor, which in the classical theory is not taken into account at all.

The third factor, which imposes serious limitations on the range of application of the *Euler method*, consists in the following. As far back as 1928–1929 Nikolai[3], while investigating one of the problems of elastic stability of a bar in torsion, discovered that the Euler method led to a paradoxical result. In the problem he was investigating there were, in general, no forms of equilibrium close to the undisturbed (rectilinear) form, and it appeared that the rectilinear form of the bar would remain stable for all values of the torque. This

---

1. Vol'mir, A. S. *Thin Plates and Shells* (Tonkie plastinki i obolochki) Gostekhizdat, 1956.

2. Shanley, F. R. Inelastic column theory. *J. Aeronaut. Sci.* **14**, 261 (1947).

3. Nikolai, E. L. On the stability of the rectilinear form of equilibrium of a bar in compression and torsion. *Izv. Leningr. politekhn. in-ta* **31** (1928); On the problem of the stability of a bar in torsion. *Vestn. prikl. matem. mekh.* **1** (1929). See also Nikolai, E. L. *Studies in Mechanics* (Trudy po Mekhanike) Gostekhizdat, 1955.

result was, however, interpreted correctly as an indication that the *Euler method* was inapplicable to that particular problem, and that it should be replaced by the more general method of investigating stability—the "method of small oscillations". It was later discovered that an important part is played here by the existence of a potential of the external forces.[1] The *Euler method* is applicable if the external forces have a potential (i.e. if they are conservative forces), and in general is not applicable if they do not.

The basic method of investigating non-conservative problems in the theory of elastic stability is the *dynamic method*, which is based on the investigation of the oscillations of the system close to its position of equilibrium. This draws the theory of elastic stability closer to the general theory of stability of motion and its applications in the theory of automatic control, in the hydromechanics of a viscous liquid and in other fields of mechanics and engineering. The *Euler method*, which reduces the problem to an analysis of the bifurcation of the forms of equilibrium of the system, can be looked upon as a particular case of the dynamic method.

The present book is devoted entirely to the study of non-conservative problems of the theory of elastic stability. Aerodynamic and hydrodynamic loads, the forces acting on parts of turbines and electric machines, and the loads induced in parts and linkages of automatic control systems are in most cases non-conservative forces. The classical theory of elastic stability developed primarily as a result of the requirements of industrial, transportation and civil engineering construction. The traditional loads of the classical theory of elastic stability are forces which have a potential, usually caused by gravitational effects (i.e. "dead" loads). As regards the non-conservative problems of elastic stability, their main interest lies in the development of present-day mechanical, aeronautical and rocket engineering.

The general questions of the stability of elastic systems under the action of non-conservative forces have been discussed in works by Ziegler and the author.[1] The discussions published in the earlier papers by Reut and B. L. Nikolai[2] are of some interest. Of the works devoted to particular problems, apart from those of E. L. Nikolai[3] indicated above, we would indicate those by Dzhanelidze, Shashkov, Deineko and Leonov, Morris, Ziegler, Troesch, Beck and Pflüger.[4] There is also a very wide range of literature on the stability of rapidly rotating shafts.[5] A considerable amount of work has also been published on the important problems of the stability of elastic elements in a flow of gas[6], in which more and more attention is now being devoted to the problem of stability of plates and shells in supersonic flow.[1] A number of problems, treated, however, from the point of view of the mechanics of systems with a finite number of degrees of freedom, are considered in the book by Rocard.[2] Some questions of stability of structures in the case of the external and internal forces having no potential are discussed in the book by Hoff.[3] The sections that follow will be devoted to an elementary introduction to the fundamentals of the subject.

---

[1] Of recent works on this subject those by Ziegler should be mentioned: ZIEGLER, H. Die Stabilitätskriterien der Elastomechanik, *Ing.-Arch.* **20**, No. 1 (1952); ZIEGLER, H. On the concept of Elastic Stability. Vol. IV, pp. 351–403, Academic Press, New York 1956.

---

[1] BOLOTIN, V. V. Questions of the general theory of elastic stability. *PMM*, **20**, No. 4 (1956); *Some problems of the theory of elastic stability*. Trans. Third All-Soviet Math. Conf. Vol. **1**, Akad. Nauk USSR, 1956; *Dynamic Stability of Elastic Systems* (Dinamicheskaia ustoichivost' uprugikh sistem), Gostekhizdat, 1956; On the vibrations and stability of bars under the action of non-conservative forces *Sb. "Kolebaniia v turbomashinakh"*, Akad. Nauk USSR (1959).

[2] REUT, V. I. On the theory of elastic stability. *Trans. Odessk. in-ta inzh. grazhd. i komm. str-va*, No. 1, 1939; NIKOLAI, B. L. On the criterion of stability of elastic systems (*ibid.*). The ideas of B. L. Nikolai are close to the modern theory.

[3] NIKOLAI, E. L. See reference on p. 2.

[4] DZHANELIDZE, G. Iu. On the stability of a bar under the action of a follower force. *Trans. Leningr. politekhn. in-ta*, No. 192 (1958); SHASHKOV, I. E. On the stability of the rectilinear form of equilibrium of a drilling shaft. *Inzh. sborn.* **1**, No. 1 (1941); SHASHKOV, I. E. On the stability of a prismatic bar of arbitrary section in compression and torsion. *Inzh. sborn.* 7 (1950); DEINEKO, K. S., and LEONOV, M. Iu. Dynamic method of investigating the stability of a compressed bar. *PMM*, No. 6 (1955); MORRIS, J. Torque and flexural stability of a cantilever. *Aircraft Engng.* **23**, No. 274 (1951); ZIEGLER, H. Stabilitätsprobleme bei geraden Stäben und Wellen, *Zeitschr. angew. Math. Phys.*, No. 4 (1951); TROESCH, A. Stabilitätsprobleme bei tordierten Stäben und Wellen. *Ing.- Arch.* **20**, No. 4 (1952); BECK, M. Die Knicklast des einseitig eingespannten, tangential gedrückten Stabes. *Z. angew. Math. Phys.* **3**, No. 3 (1952); PFLÜGER, A. Zur Stabilität des tangential gedrückten Stabes. *Z. Angew. Math. Mech.* **35**, No. 5 (1955).

[5] KAPITSA, P. L. Stability and transition through critical speeds of high-speed rotors in the presence of friction. *Zh. tekhn. fiz.* **9**, No. 2 (1939); DIMENTBERG, F. M. *Bending Oscillations of Rotating Shafts* (Izgibnye kolebaniia vrashchaiushchikhsia valov) Akad. Nauk USSR, 1959; BOLOTIN, V. V. *Investigation of self-oscillations of a flexible shaft induced by internal friction and related factors. Nauchn. dokl. vyssh. shkoly, ser. "Mashinostroenie i priborostroenie"* No. 4 (1958); BOLOTIN, V. V. *Non-linear vibrations of shafts beyond their critical speeds of rotation.* Collection "Problemy prochnosti v mashinostroenii" (Problems of strength in machine construction), No. 1, Akad. Nauk USSR, 1958.

[6] See, for example, the book *Aeroelasticity* by R. L. BISPLINGHOFF, H. ASHLEY and R. L. HALFMAN, Addison-Wesley, Cambridge, Mass. 1955.

2. Longitudinal Bending of an Axially Compressed Bar

To explain the reasons why in a number of cases the formulation of the stability problem differs from the classical Euler formulation,

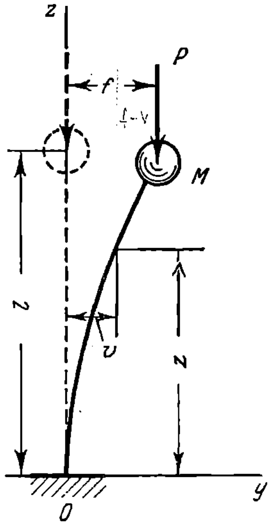


Fig. 1

---

1. Bolotin, V. V. Vibrations and stability of an elastic cylindrical shell in compressible fluid flow. *Inzh. sborn.* **24** (1956); Hedgepeth, J. On the flutter of panels at high Mach numbers. *J. Aeronaut. Sci.* **23**, No. 6 (1956); Movchan, A. A. On the stability of a panel moving in gas. *PMM*, **21**, No. 2 (1957); Bolotin, V. V. On critical velocities in the non-linear theory of aeroelasticity. *Nauchn. dokl. vyssh. shkoly, ser. "Mashinostroenie i priborostroenie"*, No. 3 (1958); Bolotin, V. V., Gavrilov, Iu. V., Makarov, B. P. and Shveiko, Iu. Iu. Non-linear problems on the stability of plane panels at high supersonic velocities. *Izv. Akad. Nauk USSR*, OTN, No. 3 (1959); Bolotin, V. V. Non-linear flutter of plates and shells. *Inzh. sborn.* **28** (1960).

2. Rocard, Y. *Dynamic Instability.* (English translation) Crosby Lockwood, London 1957.

3. Hoff, N. J. Buckling and Stability, *J. Roy. Aeronaut. Soc.* (Jan. 1954) (Aeronautical Reprints No. 123).

we shall start by considering the problem, well-known from elementary courses in strength of materials, of the longitudinal bending of a bar (Fig. 1).

We shall find the minimum value of the compressive force P for which, in addition to the rectilinear form of equilibrium, a slightly curved form first becomes possible. Having chosen the axes of coordinates as shown in Fig. 1, we can derive a differential equation for the elastic line of the slightly bent bar:

$$EJ\frac{d^2 v}{dz^2} = P(f - v). \tag{1}$$

Here EJ is the bending stiffness of the bar, f is the deflection at the end of the bar. Making use of the familiar designation

$$\frac{P}{EJ} = k^2, \tag{2}$$

we can write equation (1) in the form

$$\frac{d^2 v}{dz^2} + k^2 v = k^2 f.$$

Its general solution is

$$v = C_1 \sin k z + C_2 \cos k z + f. \tag{3}$$

In order to find the three constants C_1, C_2 and f we have the three boundary conditions: $v(0) = 0$, $v'(0) = 0$ and $v(l) = f$. Substituting expression (3) into the boundary conditions, we obtain the three relations:

$$\left.\begin{aligned} C_2 + f &= 0, \\ k C_1 &= 0, \\ C_1 \sin k l + C_2 \cos k l &= 0, \end{aligned}\right\} \tag{4}$$

which constitute a system of three linear algebraic equations in the three constants C_1, C_2 and f. The trivial solution of this system $C_1 = C_2 = f = 0$ corresponds to the rectilinear form of equilibrium of the compressed bar. Following the procedure adopted by Euler, we shall try to find the conditions under which, in addition to the rectilinear form of equilibrium, slightly curved forms close to the rectilinear form are possible. In order that the homogeneous system (4) has a non-trivial solution its determinant must vanish, i.e.

$$\begin{vmatrix} 0 & 1 & 1 \\ k & 0 & 0 \\ \sin k l & \cos k l & 0 \end{vmatrix} = 0.$$

It follows that $\cos k\,l = 0$, and the minimum value of the force corresponding to this condition is $\cos kl$ [illegible] $kl = \frac{\pi}{2}$

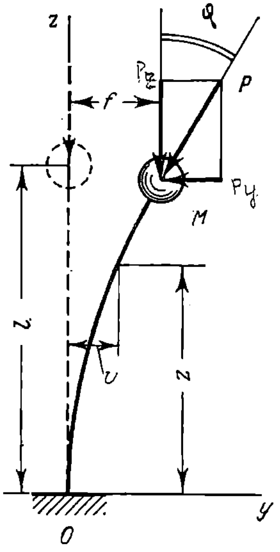
$$P_* = \frac{\pi^2 E J}{4 l^2}, \qquad (5)$$

i.e., it is equal to the Euler value of the compressive force.

In the classical theory of elastic stability it is considered to be self-evident that when the force P reaches the minimum value at which, in addition to the undisturbed (rectilinear) form of the bar, other very close curvilinear forms become possible, this value is the critical value. This definition of the critical force is not only the basis of methods which employ directly the differential equations of equilibrium, but is also the basis of a number of other methods.

3. Bar under the Action of a "Follower" Force. Euler's Method

We can extend the *Euler method* to the problem of the stability of a bar under the action of a compressive force which after deformation rotates together with the end section of the bar and at all times remains tangential to its deformed axis (Fig. 2). We shall leave open the question of how such a "*follower*" force is realized.



$P_\eta = P \sin\varphi \approx P\varphi$

$P_\zeta = P \cos\varphi \approx P$

Fig. 2

By virtue of the assumed smallness of the deflections, $P_x \approx P$, $P_y \approx P\varphi$, where φ is the angle of rotation of the end section and $\varphi \approx v'(l)$.

The equation of the elastic line of the slightly bent bar is given by

$$EJ\frac{d^2v}{dz^2} = P(f - v) - P\varphi(l - z), \tag{6}$$

and its general solution, making use of the definition (2), can be written in the form

$$v = C_1 \sin kz + C_2 \cos kz + f - \varphi(l - z). \tag{7}$$

There are four boundary conditions in this particular problem: $v(0) = 0$, $v'(0) = 0$, $v(l) = f$, $v'(l) = \varphi$. If we substitute Expression (7) into these conditions we obtain the four expressions

$$\begin{aligned} C_1 \cos kl - C_2 \sin kl &= 0, \\ C_2 + f - \varphi l &= 0, \\ kC_1 + \varphi &= 0, \\ C_1 \sin kl + C_2 \cos kl &= 0, \end{aligned}$$

the determinant of which is

$$\Delta = \begin{vmatrix} 0 & 1 & 1 & -l \\ k & 0 & 0 & 1 \\ \sin kl & \cos kl & 0 & 0 \\ \cos kl & -\sin kl & 0 & 0 \end{vmatrix} = -1$$

i.e. is non-zero for all values of P. This means that there are no values of P for which there can exist curvilinear forms of equilibrium of the bar close to the undisturbed (rectilinear) form.

Pflüger and Feodos'ev,[1] who were first to investigate this problem, came to the conclusion that a bar compressed by a "follower" force cannot in general lose stability. In order to establish whether such a conclusion is correct, we must take a closer look at the fundamental concept underlying the Euler method, which relates it to the generally accepted concepts of stability of equilibrium (and of motion).

---

[1] Pflüger, A. Stabilitätsprobleme der Elastostatik, p. 217, Springer-Verlag, Berlin 1950; Feodos'ev, V. I. *Selected Problems and Questions in Resistance of Materials* (Izbrannye zadachi i voprosy po soprotivleniiu materialov) p. 165, Gostekhizdat, 1953.

Expressed in technical terms the concept of stability may be stated as follows. The form of equilibrium is said to be stable if a slight disturbance causes a small deviation of the system from the considered configuration of equilibrium, but by decreasing the magnitude of the disturbance the deviation can be made as small as required. On the other hand, the equilibrium is said to be unstable if a disturbance, however small, causes a finite deviation of the system from the considered form of equilibrium.

It will readily be seen that the concept of stability is not in effect implicit in the *Euler method* of the theory of elastic stability. It is possible, of course, that if there is a form of equilibrium close to the initial (undisturbed) form, the latter might become unstable by being replaced by another form of equilibrium. However, the existence of a branch point is neither a necessary nor a sufficient condition for loss of stability. The initial form of equilibrium may cease to be stable not only at branch points, but also at so-called limit points.[1] On the other hand, cases are known where the existence of branch points does not lead to instability of the initial form of equilibrium.[2] The most essential fact to bear in mind is that in using the *Euler method*, i.e. in considering only a number of forms of equilibrium, we automatically exclude from our analysis any possible forms of motion. The initial form of equilibrium may remain unstable, having been replaced by some form of motion, whereas if we use the *Euler method* we consider only the possible forms of equilibrium close to the initial form.

The complete solution can be obtained by assuming that the disturbance depends on time and by deriving equations which will enable the variation of the disturbance with time to be taken into account. Roughly speaking the form of equilibrium under investigation will be stable if all the disturbances are limited in time.

4. Stability with Respect to Small Disturbances

Let us return to the elementary problem of the stability of a bar compressed by a force acting in a fixed direction. We shall derive the equation of small oscillations of the bar about its rectilinear position, assuming for simplicity that the distributed mass of the bar is negligible compared with the mass M concentrated at the end (Fig. 1), and that resistance forces can be ignored.

---

[1] Bolotin, V. V. Non-linear theory of elasticity and stability in its widest sense. *Raschety na prochnost'*, No. 3, Mashgiz (1958).

[2] Ishlinskii, A. Iu., Malashenko, S. V. and Temchenko, M. E. On the branching of stable positions of dynamic equilibrium of a mechanical system. *Izv. Akad. Nauk USSR*, OTN, No. 8 (1958).

The equation of small oscillations is of the form

$$EJ\frac{\partial^2 v}{\partial z^2} = P(f - v) - M(l - z)\frac{d^2 f}{dt^2}, \tag{8}$$

where $v(z, t)$ is the dynamic deflection at each point and $f(t)$ is the deflection at the end. Equation (8) is satisfied by the expressions

$$v(z, t) = V(z)e^{i\Omega t}, \qquad f(t) = Fe^{i\Omega t}, \tag{9}$$

where Ω is some constant to be found. If this constant is real, then the bar performs harmonic oscillations with frequency Ω about its rectilinear position. In this case we can assume (although we shall see later that this is not quite so) that the rectilinear form of the bar is stable. If Ω proves to be a complex number and if its imaginary part is negative, then this would correspond to disturbances which become infinitely large (as a result of linearizing the problem) with time. For, if $\Omega = \alpha + i\beta$, then

$$v(z, t) = V(z)e^{(i\alpha-\beta)t},$$

and if $\beta < 0$ an exponentially increasing term appears in the expression for $v(z, t)$.

If we now substitute (9) into Equation (8) and use the definition (2), we obtain

$$\frac{d^2 V}{dz^2} + k^2 V = k^2 F + \frac{M\Omega^2}{EJ}(l - z)F.$$

The general solution of this equation is

$$V(z) = C_1 \sin kz + C_2 \cos kz + F + \frac{M\Omega^2}{k^2 EJ}(l - z)F,$$

for which the boundary conditions remain as before: $V(0) = 0$, $V'(0) = 0$, $V(l) = F$. Making use of these conditions, we obtain a system of three linear algebraic equations in C_1, C_2 and F, and if we equate to zero their determinant we obtain the "frequency equation"

$$\begin{vmatrix} 0 & 1 & 1 + \dfrac{M\Omega^2 l}{k^2 EJ} \\ k & 0 & -\dfrac{M\Omega^2}{k^2 EJ} \\ \sin kl & \cos kl & 0 \end{vmatrix} = 0.$$

Whence,

$$\Omega = \pm \sqrt{\left[\frac{P}{M l} \frac{\cos k l}{\dfrac{\sin k l}{k l} - \cos k l}\right]}. \tag{10}$$

Provided the expression under the root is positive, both frequencies are real, i.e. the rectilinear form of the bar is stable (in the sense indicated above). The frequencies vanish when $\cos k l = 0$ and become purely imaginary with further increase in the force P, when one of them has a negative imaginary part. The condition $\cos k l = 0$ corresponds to transition from the stable rectilinear form to an unstable form, and the critical value of the force P apparently coincides with the Euler value (5).

Thus, in the problem of the stability of a bar compressed by a force acting in a fixed direction, the *Euler method* and the dynamic method give identical results.

The difference between Euler and dynamic [illegible] is only from the [illegible] of mass

5. CRITICAL VALUE OF "FOLLOWER" FORCE

We shall now apply the dynamic method to the problem of the stability of a bar compressed by a "follower" force (see Fig. 2). The equation of small oscillations of a bar about its position of equilibrium in this case is

$$E J \frac{\partial^2 v}{\partial z^2} = P(f - v) - P \varphi(l - z) - M \frac{d^2 f}{d t^2}(l - z). \tag{11}$$

By making the substitutions

$$v(z, t) = V(z) e^{i\Omega t}, \quad f(t) = F e^{i\Omega t}, \quad \varphi(t) = \Phi e^{i\Omega t}$$

we reduce Equation (11) to the form

$$\frac{d^2 V}{d z^2} + k^2 V = k^2 F - k^2 \Phi(l - z) + \frac{M \Omega^2 F}{E J}(l - z).$$

Its solution

$$V(z) = C_1 \sin k z + C_2 \cos k z + F - \Phi(l - z) + \frac{M \Omega^2 F}{k^2 E J}(l - z)$$

must satisfy the boundary conditions $V(0) = 0$, $V'(0) = 0$, $V(l) = F$, $V'(l) = \Phi$. From this we obtain the "frequency equation"

$$\begin{vmatrix} 0 & 1 & 1 + \dfrac{M\Omega^2 l}{k^2 E J} & -l \\ k & 0 & -\dfrac{M\Omega^2}{k^2 E J} & 1 \\ \sin k l & \cos k l & 0 & 0 \\ k \cos k l & -k \sin k l & -\dfrac{M\Omega^2}{k^2 E J} & 0 \end{vmatrix} = 0,$$

the solution of which gives the formula

$$\Omega = \pm \sqrt{\left[\frac{P}{M l} \frac{1}{\dfrac{\sin k l}{k l} - \cos k l}\right]}. \tag{12}$$

As the force P increases from zero, the natural frequencies increase in magnitude; when

$$\tan k l = k l \tag{13}$$

they become infinite, and with further increase in P they become purely imaginary, their signs remaining opposite. The smallest root of equation (13) corresponds to transition from the stable rectilinear form to an unstable form. The value of this root is $k l = 4.493$, which, from (2), gives a critical value of the "follower" force of

$$P_* = \frac{20.19\, E J}{l^2}. \tag{14}$$

Thus, in the case of a bar under the action of a "follower" force, the *Euler method* leads to erroneous results: the rectilinear form of the bar for sufficiently large values of the force P proves to be unstable, although in this particular problem there cannot exist curvilinear forms of equilibrium close to the initial rectilinear form.

6. CRITICAL VALUE OF "FOLLOWER" FORCE (CONTINUED). SYSTEM WITH TWO DEGREES OF FREEDOM

The features of the problem considered in Section 5, become even more apparent if we consider a system with two degrees of freedom. We can take, for such a system, for example, a bar with two concentrated masses[1].

Let M_1 and M_2 be the masses, $f_1(t)$ and $f_2(t)$ their displacements in the direction of the axes Oy (Fig. 3). To obtain the frequency equations we choose a procedure slightly different from the one employed in Section 5. Using D'Alembert's principle we form the equations of the small oscillations of the bar about the undisturbed equilibrium configuration:

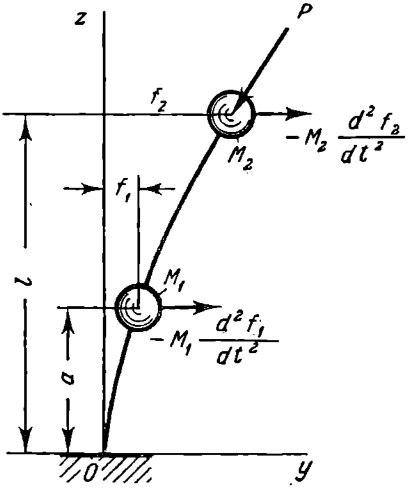


Fig. 3

$$f_1 = -\delta_{11} M_1 \frac{d^2 f_1}{dt^2} - \delta_{12} M_2 \frac{d^2 f_2}{dt^2},$$

$$f_2 = -\delta_{21} M_1 \frac{d^2 f_1}{dt^2} - \delta_{22} M_2 \frac{d^2 f_2}{dt^2},$$

where

$$\delta_{11}(P), \qquad \delta_{12}(P), \qquad \delta_{21}(P) \quad \text{and} \quad \delta_{22}(P)$$

are the corresponding displacements produced by unit transverse forces applied to the bar. These displacements are determined, obviously, taking into account the force P.

---

[1] This problem was considered by K. S. DEINEKO and M. IU. LEONOV (see Ref. on page 4).

The unit displacements are easily found from elementary considerations. For this purpose we use the equation of static bending

$$EJ\frac{d^4v}{dz^4} + P\frac{d^2v}{dz^2} = 0, \tag{15}$$

which is obtained from equation (6) as a result of repeated termwise differentiation. The equation possesses a general integral

$$v(z) = C_1 \sin kz + C_2 \cos kz + C_3 z + C_4,$$

where, as before, the notation (2) is used. Considering in succession the case of the effect of a unit force applied to the masses M_1 and M_2, and subjecting the solution to boundary conditions we obtain:

$$\begin{aligned}
\delta_{11} &= \frac{1}{kP}(\sin ka - ka\cos ka),\\
\delta_{12} &= \frac{1}{kP}[\sin kl - ka\cos kl - \sin k(l-a)],\\
\delta_{21} &= \frac{1}{kP}[\sin kl - kl\cos kl + \sin k(l-a)],\\
\delta_{22} &= \frac{1}{kP}(\sin kl - kl\cos kl).
\end{aligned} \tag{16}$$

Let us note that, in general, $\delta_{12} \neq \delta_{21}$, i.e. the reciprocity principle for displacements is not valid. We shall return to this remark in the sequel. Analogous to (9) we set

$$f_1(t) = F_1 e^{i\Omega t}, \qquad f_2(t) = F_2 e^{i\Omega t}. \tag{17}$$

As a result we obtain the "frequency equation"

$$\begin{vmatrix} M_1\delta_{11}\Omega^2 - 1 & M_2\delta_{12}\Omega^2 \\ M_1\delta_{21}\Omega^2 & M_2\delta_{22}\Omega^2 - 1 \end{vmatrix} = 0.$$

Expanding the determinant and solving the equation, we arrive at the formula

$$\Omega^2 = \frac{M_1\delta_{11} + M_2\delta_{22} \pm \sqrt{[(M_1\delta_{11} - M_2\delta_{22})^2 + 4M_1M_2\delta_{12}\delta_{21}]}}{2M_1M_2(\delta_{11}\delta_{22} - \delta_{12}\delta_{21})}. \tag{18}$$

For sufficiently small values of the force P formula (18) gives real values for the frequency of vibrations. However, as the force P reaches a certain critical value P_*, the expression under the radical becomes equal to zero and the frequency equation will have a double root. With further increase of the force P among the natural frequencies there will be a frequency which has a negative imaginary part. As may be seen from formula (17), this corresponds to unbounded increase of disturbances in time, i.e. instability of the rectilinear form of the equilibrium of the bar.

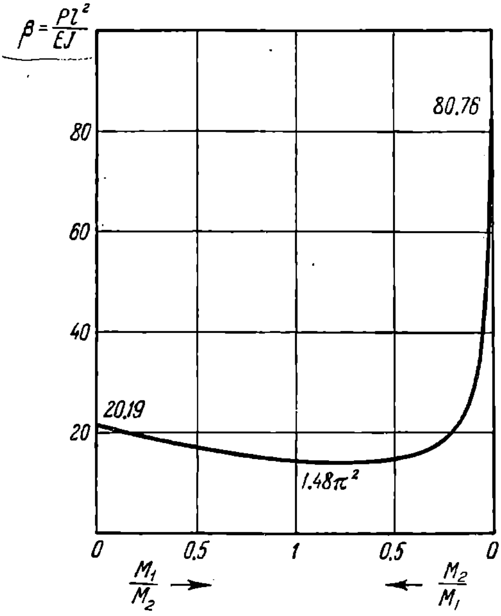


Fig. 4

The formula for the critical force may be expressed in the form

$$P_* = \frac{\beta EJ}{l^2}.$$

In Fig. 4 results of calculation are presented for the case $a = l/2$. For $M_1 = 0$ we obtain that $\beta \approx 20.19$, i.e. we are led to formula (14). With $M_2 = 0$ the critical value, as should have been expected, becomes exactly four times as large.

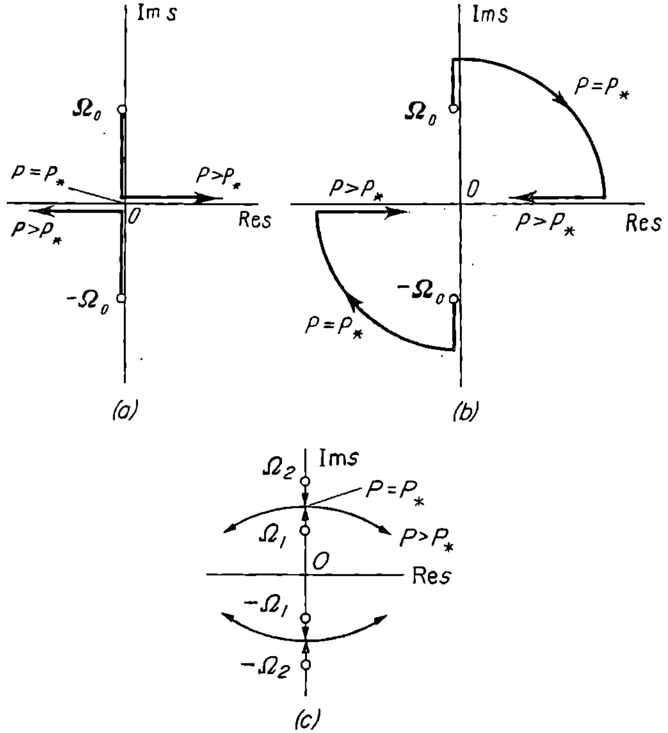
7. Discussion of Results. Potential of External Forces

In connection with the problems we have just examined, the following questions arise. Why is it that the *Euler method* is acceptable in one case, but not in the other? What is the physical meaning of the instability discovered in the second problem?

To find the answer to these questions let us examine once again the behavior of the bar in both problems after the occurrence of small deviations from the rectilinear form. Instead of the frequencies Ω, we introduce the characteristic exponents connected to the frequencies by the simple relation $s = i\Omega$. The expression for the dynamic deflection of the bar then assumes the form

$$v(z, t) = V(z)\, e^{st}. \tag{19}$$

The rectilinear form of the bar becomes unstable if one of the characteristic exponents has a positive real part.



(a) (b)

(c)

Fig. 5

The behavior of the characteristic exponents can conveniently be represented graphically, their real parts being taken as abscissae and their imaginary parts as ordinates; points corresponding to various values of P are then joined to give a continuous curve.

Figure 5 shows the variation of the characteristic exponents for the first case (a force acting in a fixed direction). When $P = 0$ we have $s = \pm i\Omega_0$, where Ω_0 is the natural frequency of oscillation of the unloaded bar. With increase in P the characteristic exponents, while remaining purely imaginary, decrease in modulus until when $P = P_*$ they vanish. The vanishing of the characteristic exponents corresponds to "neutral" equilibrium—to the existence, in addition to the initial form of equilibrium, of other infinitely close forms of equilibrium. If $P > P_*$ one of the characteristic exponents is real and positive, which corresponds to instability of the rectilinear form of equilibrium of the bar.

Whereas in the case of a force acting in a fixed direction the characteristic exponent enters the right-hand half-plane of the complex variable, passing through the value of $s = 0$, in the case of a "follower" force (Fig. 5b, 5c) it enters this half-plane without passing through this value. In the first case the *Euler method* can be used to establish the point of transition from stability to instability; in the second case it cannot.

The Euler method is applicable in cases when the external forces acting on the body are conservative, i.e. when they have a potential (since everywhere in this book we are dealing with ideally elastic bodies, the internal forces in all problems do have a potential). It can easily be shown, and this will be done at the appropriate place in the book, that when the external forces have a potential, transition from stability to instability takes place only through the value of $s = 0$, i.e. in this case the *Euler method* gives correct results. But if the external forces are not conservative, then, although in a number of problems it may give the correct solution, the *Euler method* in order to avoid errors, should be replaced by the more general *dynamic method* based on consideration of the oscillations of the system about the equilibrium position under investigation.

It is not difficult to see that in the problem under consideration the "follower" force has no potential. The indication of whether a potential of the external forces exits is provided by the independence of the work done by these forces on the manner in which the system attains its final state. Figure 6 shows three ways in which the bar can reach a state defined by a transverse deflection f and an angle of rotation φ of the end section. In case (a) (rotation through an angle φ with a subsequent displacement) the work done by the force P is obviously negative; in case (b) (displacement with a subsequent rotation) it is zero, and in case (c) (rotation through an angle $-\varphi$, a displacement and a final rotation through an angle 2φ) the work done is positive.[1]

These considerations illustrate one feature inherent in all non-conservative problems dealt with in this book. In contrast to the well-known and thoroughly studied self-excited oscillating systems, the non-conservative nature of which depends on "negative friction", in the problems considered here the coupled interaction of the degrees of freedom plays a significant part. The supply of energy to a bar subjected to a "follower" force (Fig. 4) is possible only for a particular change with time of the angle of rotation φ and of the linear displacement f. This coupled interaction becomes more obvious if we consider the small oscillations of the bar as a system with *two* degrees of freedom (see Section 6). The instability of the system is of an oscillatory nature with a definite phase relation between φ and f corresponding to the absorption of energy from outside. The case of a mass concentrated at a point considered in Section 5 should be treated as the limiting case of a system with two degrees of freedom in which the second partial frequency tends to infinity.

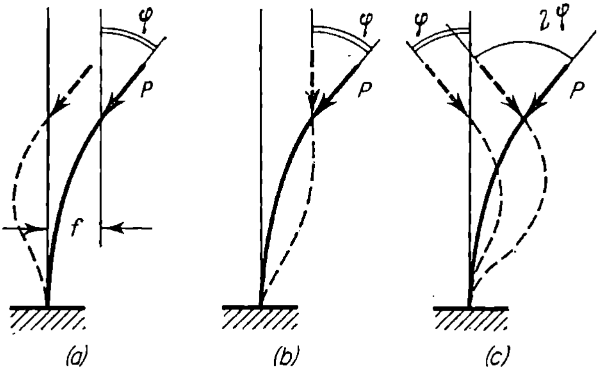


Fig. 6

8. Range of Problems Covered in the Book

In the past the development of the theory of elastic stability was related principally with the requirements of industrial, transportation and civil engineering construction. Weight forces and various other forces of gravitational origin are typical of the loads considered in the classical theory of elastic stability.

---

[1] We call attention also to the fact that the reciprocity relations for displacements are not satisfied, as indicated in discussing formulae (16).

The development of modern mechanical, aeronautical and rocket engineering has considerably widened the class of load types requiring investigation. Aerodynamic forces acting on the wing of an aircraft or on the body of a rocket are examples of non-conservative forces. So also are the forces acting on the rotor of an electric motor or on the rotor of a gas turbine. All automatic control systems are non-conservative systems. Therefore, in an examination of the elastic stability of the links and elements in automatic control, the action of the remaining links of the system is replaced by forces which are also non-conservative. It will be seen then that the study of the conditions of stability of elastic systems under the action of non-conservative forces is of particular importance in modern engineering practice.

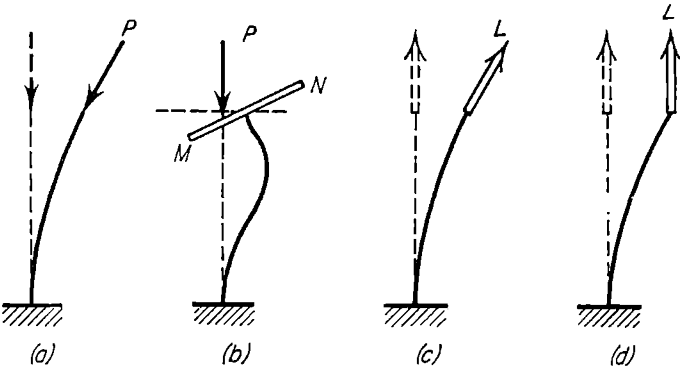


Fig. 7

In addition to the more general questions, the book deals with three types of problems. The first are problems of the stability of elastic bodies subjected to "follower" forces, i.e. to forces which during the process of loss of stability follow some particular law other than that for weight forces. The problem considered above, on the stability of a bar in compression under the action of a force, the direction of which remains tangential to the deformed axis of the bar, is the most simple problem of this type (Fig. 7a). Such a force can be produced, for example, by the reaction of a jet attached to the end of the bar (on the condition that the effects of inertia and damping can be ignored). Another non-conservative problem is the problem of the stability of a bar compressed by a force which is always directed along the initial undeformed axis (Fig. 7b). It is clear that something similar to such a force could be produced by the pressure from a jet of absolutely inelastic particles. The fact that in both cases the external forces have no potential is clear from the very manner in which they are realized. Let us still show that in the case, expressed in Fig. 7b, the work of the force P in general will depend on the manner in which the system passes from one state to the other. We note first of all, that the force in this case is not rigidly connected with the material points of the bar MN, but is sliding along this bar. Therefore the elementary work δA of the force P should be evaluated by the expression $\delta A = \boldsymbol{P} \cdot \boldsymbol{v}\, dt$, where $\boldsymbol{v}$ is the velocity vector of the material point on which the force vector $\boldsymbol{P}$ is acting at the instant of time considered[1]. Figure 8 indicates two ways in which the system can pass into a state characterized by the deflection f and the angle of rotation φ. In case *a* (displacement followed by rotation) the work is positive, whereas in case *b* (rotation followed by displacement) it is equal to zero, and so on. A further example of non-conservative problems of elastic stability is a shaft under the action of a torque whose vector remains directed along the tangent to the axis of the shaft during the process of loss of stability (Fig. 7c). It is interesting to note that

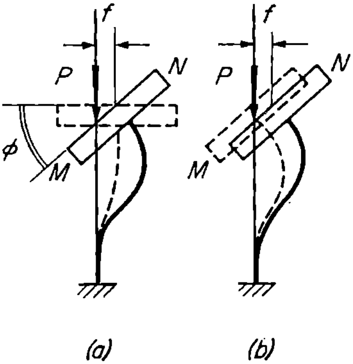


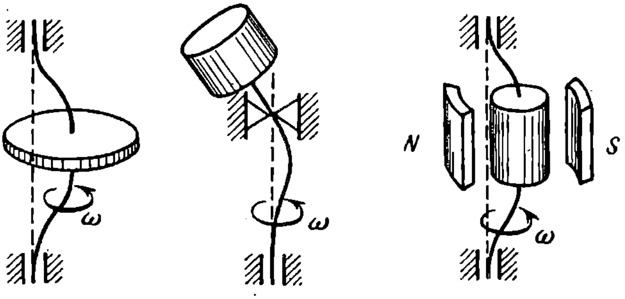
Fig. 8

---

[1] In courses on mechanics forces are considered which are rigidly connected with material points, and as a result one usually does not emphasize the difference between the displacements of the material points and the displacements of the points of application of the force. The neglect of this distinction has led to misunderstandings in several published papers.

when the direction of the vector of the torque remains fixed in space (Fig. 7d) the problem is again non-conservative. Loading conditions similar to the last two examples are often encountered in certain types of turbines.

The second category of problems is encountered in the investigation of the links and objects in automatic control systems. The most important of these are problems on the stability of flexible shafts with a controlled number of revolutions. It is a well-known fact that even ideally balanced shafts working beyond their critical number of revolutions often have a tendency to develop dangerous bending vibrations. There are a number of factors which might cause such vibrations: internal friction in the material of the shaft, friction between a disk and the surrounding medium, the effect of the oil layer in bearings, etc. The basic reason, however, lies in the non-conservatism of the system. If the drive maintains a constant angular velocity irrespectively of the behavior of the shaft, then under certain conditions part of the energy is adsorbed in lateral oscillations; internal friction and similar factors then act simply as transmitters of energy (Fig. 9a). Similar phenomena are observed in centrifuges when partially filled with a liquid (Fig. 9b) and also in ferro-magnetic or conducting rotors situated in a magnetic field (Fig. 9c).



(a) (b) (c)

Fig. 9

The third category of problems is encountered in aeronautical engineering. These problems are associated mainly with the familiar and well-studied problem of wing flutter, which is a typical non-conservative problem of elastic stability (Fig. 10a). If we apply the *Euler method* to the problem of wing stability we can find only the critical speed of divergence (i.e. the speed at which static overtwist of the wing occurs). In order to determine the critical flutter velocity it is necessary to consider small oscillations of the wing about its position of equilibrium. Recently a considerable amount of attention has been devoted to problems in the stability of plates (Fig. 10b) and shells (Fig. 10c) situated in a flow of gas (so-called problems of "panel flutter"). These problems, especially in cases of high supersonic velocities, have much in common with the first class of problems. The difference between them lies more in the degree of schematization of the forces acting on the elastic body than in any other feature.

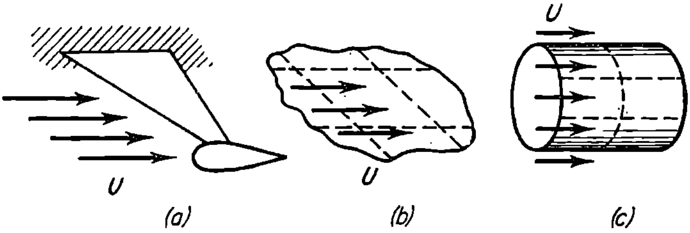


Fig. 10

All of the problems listed above, in spite of their considerable physical disparity, are mathematically very similar, since they all lead to non-self-adjoint boundary-value problems. This is the essential difference between the types of problems considered here and those covered by the classical theory of elastic stability, in which, by virtue of the existence of potential in the external forces, all the problems are self-adjoint.

The methods of investigating non-conservative problems of the theory of elastic stability can be divided into two groups. The first group consists of methods based on a direct analysis of the differential equations of motion of the elastic body. By convention these are called "exact" methods. These methods usually entail considerable computational difficulties and for this reason only some of the most simple problems have been solved. Moreover, the theory of non-self-adjoint operators is not one of the very well developed branches of mathematics which have effective methods for solving practical problems. The other group includes approximate methods, the basis of which is to replace the elastic body by some equivalent system with a finite number of degrees of freedom and then to analyze this equivalent system. Although in the majority of actual problems the validity of such a replacement is not in doubt, a strict mathematical justification has yet to be found. Some general justification can be claimed by comparing well-known exact results with the results of an approximate solution. In the following chapters we shall use methods belonging to both groups, and where an exact solution exists, a comparison of the exact and approximate solutions will be given. In non-linear problems it is necessary to rely exclusively on approximate methods.

CHAPTER 1

GENERAL PRINCIPLES

1.1. INTRODUCTORY REMARKS

The basic elements with which the theory of elastic stability is concerned, and which give stability problems a practical meaning, are bars, frameworks, plates and shells. Nevertheless, in formulating the general questions of the theory of elastic stability it is expedient to start from the general equations for an elastic body. The equations for the more common elements can always be derived from the general equations of the theory of elasticity by introducing appropriate kinematic and static hypotheses and applying variational principles. Consequently, by starting from the general equations of the theory of elasticity, we gain advantages in two respects. First, we are able to consider the problem of elastic stability as a whole, without dividing it into separate parts (stability of bars and frameworks, stability of plates, stability of shells). Secondly, we are able to derive the equations for specific elements by a single method from the general expressions, instead of using particular (although, perhaps, more familiar) techniques. For this reason, when considering general questions, we shall always start from the equations of the theory of elasticity for an arbitrary elastic body.

An investigation of the stability of equilibrium of an elastic body with respect to small disturbances (i.e. stability "in the small") leads, as is wellknown, to linear differential equations. However, these equations differ from the equations of the classical theory of elasticity by the presence of additional terms containing parameters to the accuracy of which the external loading is specified. In order to obtain these terms it is necessary to differentiate between the geometry of the initial undisturbed state, the stability of which is the subject of investigation, and the geometry of the other very close states. In the classical theory of elasticity this difference is, of course, not taken into account. In order to derive the equations which describe the behavior of small disturbances for a given form of equilibrium, it is necessary to start from the equations of the non-linear theory of elasticity.

Furthermore, if we do base our analysis on the equations of the classical theory of elasticity, then by virtue of Kirchhoff's uniqueness theorem, we conclude that for given loads and boundary conditions only one form of equilibrium is possible. The presentation of the problem of elastic stability, however, presupposes the possibility of the existence of forms of equilibrium (or motion) other than the undisturbed form. If we investigate stability of equilibrium in the Euler sense, we try to find values of the loading parameter for which, in addition to the undisturbed form of equilibrium, there can exist other very close forms.

The non-linearity of the equations of the theory of elasticity can have both a geometrical and a physical origin. Geometrical nonlinearity is associated with the necessity to distinguish between the coordinates of the initial and final states, and also with the necessity to use the complete expressions for the components of the strain tensor. Physical non-linearity stems from the non-linear relation between the stress and strain components, which must be taken into account in a number of problems.

In their application to problems of elastic stability it is possible to use a simplified version of the equations of the non-linear theory of elasticity. The question of the admissibility of such simplifications is dealt with in detail in the book by Novozhilov.[1] It should be pointed out here that, since we are dealing with the problem of elastic stability, we can confine our attention to the case of small strains but large displacements. Or, more precisely, in deriving the stability equations, we need not take into account the difference between the lengths, areas and volumes before and after deformation. This assumption obviously does not go beyond the framework of normal theories of applied science. For instance, in the problem of the longitudinal bending of a bar in compression it is of no practical significance to take into account the change in length or cross-sectional area of the rod during the loading process, since the associated error is extremely small. This statement does not apply to rubber or similar materials, which are outside the scope of the present book.

[1] NOVOZHILOV, V. V. *Foundations of the Non-linear Theory of Elasticity* (Osnovy nelineinoi teorii uprugosti), Gostekhizdat, 1948. (English translation published by Graylock Press, 1953).

We shall commence our study of the problems of elastic stability with a summary of the non-linear theory of elasticity. Although it is natural to adopt the methods of tensor calculus, we shall not do so. We shall only use certain definitions which combine the properties of conciseness and clarity. With a few exceptions we shall use a system of rectangular Cartesian coordinates, denoting the coordinates by x_1, x_2, x_3 (instead of the usual x, y, z). The stresses acting on the faces of a rectangular parallelepiped (Fig. 11) will be denoted by $\sigma_{11}, \sigma_{12}, \ldots, \sigma_{33}$, using the same notations for normal and shear stresses. The well-known equations of equilibrium can then be written in the form (X_1, X_2, X_3 being the components of the body forces):

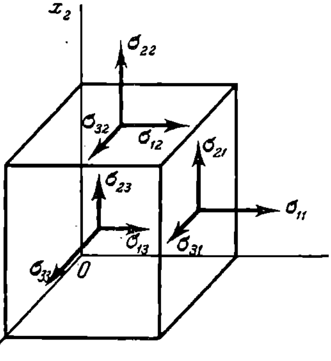


Fig. 11

$$\left.\begin{aligned}
\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}+X_1&=0,\\
\frac{\partial\sigma_{21}}{\partial x_1}+\frac{\partial\sigma_{22}}{\partial x_2}+\frac{\partial\sigma_{23}}{\partial x_3}+X_2&=0,\\
\frac{\partial\sigma_{31}}{\partial x_1}+\frac{\partial\sigma_{32}}{\partial x_2}+\frac{\partial\sigma_{33}}{\partial x_3}+X_3&=0.
\end{aligned}\right\}\qquad(1.1)$$

Finally, we shall follow the rule of summation with respect to dummy indices. This rule states that in the product of two or more factors, if an index occurs twice (such an index is called a "dummy" index), then summation is understood with respect to this index, the summation sign being omitted for the sake of brevity. For example, Equations (1.1) can now be written in the form

$$\frac{\partial \sigma_{ik}}{\partial x_k} + X_i = 0. \tag{1.2}$$

The presence of the "free" index i means that there is not one, but three equations for values of $i = 1, 2, 3$.

1.2. Finite Strain

A number of books[1] have been written on the non-linear theory of elasticity (or, as it is sometimes called, the theory of finite strain). We shall outline here only the parts of this theory essential for the understanding of the exposition which follows.

The main feature which distinguishes the theory of finite strain from the classical theory of elasticity is the necessity to differentiate between the geometries of the undeformed and deformed states. The deformed state can be described in several ways. We can, for example, use a fixed system of coordinates unattached to the body (Fig. 12a). In this case the position of a point M before deformation will be defined by three coordinates x_k, and its position after deformation by three coordinates $\tilde{x}_k = x_k + u_k$ where u_k are the components of the displacement vector. (Here and in the future the symbol $\sim$ is used to denote the deformed state). Alternatively, we could use a coordinate system which deforms together with the body. In the undeformed state this system is Cartesian and in the deformed

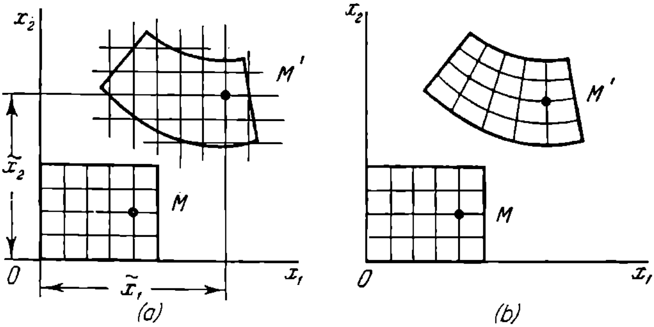


Fig. 12

---

[1] Novozhilov, V. V. See reference on page 26; Mournaghan, F. *Finite Deformation of an Elastic Solid*, Wiley, New York 1951; Gol'denblatt, I. I. *Some Questions of the Mechanics of Deformable Media* (Nekotorye voprosy mekhaniki deformiruemykh sred) Gostekhizdat, 1955.

state it becomes, in general, a curvilinear system (Fig. 12b). In this case the coordinates of a point M are the same before and after deformation, i.e. $\tilde{x}_k = x_k$. The first method is widely used in hydromechanics and is called the *Euler method.* The second is called the Lagrange method and for a number of reasons is to be preferred in the theory of elasticity. In future we shall use the Lagrange variables.

The second difference between the classical and the non-linear theories of elasticity is in the choice of components of the deformed state. In the classical theory the components of the strain tensor are defined by the expressions.

$$e_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \tag{1.3}$$

and have the meaning of relative extensions (for $i = k$) and half the shear angle (for $i \neq k$). In the case of finite strain this interpretation no longer holds. It would appear that in this case the exact expressions for linear and shear strains should be taken as the deformation characteristics. However, these expressions do not satisfy the conditions of tensor invariance and are therefore most unsuitable for a general theory. The role of deformation characteristics in the non-linear theory of elasticity is played by the components of the tensor of finite strain

$$\varepsilon_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} + \frac{\partial u_j}{\partial x_i}\,\frac{\partial u_j}{\partial x_k}\right), \tag{1.4}$$

the mechanical interpretation of which will be clear from the following considerations.

The difference between the square of the length of an element ds in any direction after and before deformation is

$$d\tilde{s}^2 - ds^2 = d\tilde{x}_j^2 - dx_j^2.$$

Here

$$d\tilde{x}_j = dx_j + \frac{\partial u_j}{\partial x_i}\,dx_i;$$

in accordance with the summation rule dx_j^2 can be expanded into the form

$$dx_j\,dx_j = dx_1^2 + dx_2^2 + dx_3^2.$$

We then find

$$d\tilde{s}^2 - ds^2 = 2\varepsilon_{ik}\,dx_i\,dx_k,$$

where ε_{ik} is defined by Formula (1.4). Expression (1.4) differs from (1.3) in that it contains non-linear terms.

1.3. Equilibrium Equations and Boundary Conditions

Considering now the equations of equilibrium and the boundary conditions, we note that in the case of finite strain even the definition of the stresses becomes ambiguous. In addition to the fact that there are various coordinate systems which can be used, the stress components can be defined in different ways; namely these components could be referred to areas after or before deformation, could be decomposed in various directions, etc. For this reason the equations of the non-linear theory of elasticity can have a completely different form, and to transfer from one form to another is not at all easy[1]. For our purposes we shall use the equations of equilibrium in the form which was under discussion as far back as the classical investigations of Kirchhoff and Boussinesq, but which finally became established as a result of work carried out by Trefftz in the field of elastic stability.[2]

We shall begin by deriving these equations on the basis of the principle of virtual displacements. We define Φ as the internal energy per unit volume in the undeformed state. Φ can be looked upon as a function of state, which, if we ignore thermal effects, depends only on the components of the strain tensor:

$$\Phi = \Phi(\varepsilon_{11}, \varepsilon_{12}, \ldots, \varepsilon_{33}). \tag{1.5}$$

The total energy stored in the body is evidently

$$U = \int\limits_V \Phi(x_1, x_2, x_3)\, dV, \tag{1.6}$$

where the integration is taken over the volume V of the undeformed body.

Application of the principle of virtual displacements leads to the expression

$$\delta U - \int\limits_V X_i\, \delta u_i\, dV - \iint\limits_S p_i\, \delta u_i\, dS = 0, \tag{1.7}$$

---

[1] An analysis of the various forms of the equations and boundary conditions can be found in the article: Galimov, K. Z. Equilibrium equations of the theory of elasticity in the case of finite displacements, and their application to the theory of shells. *Izv. Kazansk. fil. Akad. Nauk USSR, ser. fiz.-mat. i. tekh. nauk*, No. 1 (1948).

[2] Trefftz, E. Über die Ableitung der Stabilitätskriterien des elastischen Gleichgewichts aus Elastizitätstheorie der endlichen Deformationen. *Verhandl. III. Intern. Kongr. techn. Mech.* **3**, Stockholm 1930.

where δu_i are kinematically admissible variations of the displacements, p_i are surface forces acting on the external surface S of the body. The forces X_i and p_i are referred, respectively, to unit volume and area before deformation.[1]

Let us evaluate the variation δU. Noting that

$$\varepsilon_{ik} = \tfrac{1}{2}(v_{ik} + v_{ki} + v_{ji}\, v_{jk}), \tag{1.8}$$

where for brevity we have used the notation

$$v_{ik} = \frac{\partial u_i}{\partial x_k}, \tag{1.9}$$

we can write (1.5) in the form

$$\Phi = \Phi(v_{11}, v_{12}, \ldots, v_{33}).$$

Then after making transformations analogous to those made in the derivation of the Euler–Ostrogradskii[2] variational equations, we find

$$\delta U = \int\limits_V \frac{\partial \Phi}{\partial v_{ij}} \frac{\partial(\delta u_i)}{\partial x_j}\, dV = \\ = \int\limits_V \frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial v_{ij}} \delta u_i\right) dV - \int\limits_V \frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial v_{ij}}\right) \delta u_i\, dV.$$

But according to the Gauss–Ostrogradskii formula,

$$\int\limits_V \frac{\partial A_j}{\partial x_j}\, dV = \iint\limits_S A_j n_j\, dS \tag{1.10}$$

(here n_j are the components of the unit vector of the exernal normal to the surface S; the conditions imposed on the components of A_j and the surface S in this case are satisfied). Thus

$$\delta U = \iint\limits_S \frac{\partial \Phi}{\partial v_{ij}} \delta u_i\, n_j\, dS - \int\limits_V \frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial v_{ij}}\right) \delta u_i\, dV.$$

---

[1] Here it is assumed that

$$\tilde{X}_i\, d\tilde{V} = X_i\, dV, \qquad \tilde{p}_i\, d\tilde{S} = p_i\, dS$$

although, by virtue of the assumption of smallness of changes of area and volume this stipulation is of no real significance.

[2] See, for example, the book: COURANT, R and HILBERT, D. *Methods of Mathematical Physics*. Interscience Publishers, New York and London 1953.

Equation (1.7) now becomes

$$\int\limits_V \left[\frac{\partial}{\partial x_j}\left(\frac{\partial \Phi}{\partial v_{ij}}\right) + X_i\right]\delta u_i \, dV - \iint\limits_S \left(\frac{\partial \Phi}{\partial v_{ij}} n_j - p_i\right)\delta u_i \, dS = 0.$$

In view of the arbitrariness of the variations we can equate terms in δu_i to zero and thus obtain the equations of equilibrium and the boundary conditions on S:

$$\left.\begin{aligned} \frac{\partial}{\partial x_j}\frac{\partial \Phi}{\partial v_{ij}} + X_i = 0, \\ \frac{\partial \Phi}{\partial v_{ij}} n_j = p_i. \end{aligned}\right\} \tag{1.11}$$

The equations and boundary conditions (1.11) coincide formally with the corresponding expressions in the classical theory of elasticity

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = 0, \qquad \sigma_{ij} n_j = p_i,$$

if we make the substitution

$$\sigma_{ij} = \frac{\partial \Phi}{\partial v_{ij}}.$$

However, only in the linear approximation does the tensor $\partial\Phi/\partial v_{ij}$ coincide with the stress tensor in the usual sense. In general it is not even symmetrical.

Equations (1.11) can be slightly rearranged by making the substitution

$$\frac{\partial \Phi}{\partial v_{ij}} = \frac{\partial \Phi}{\partial \varepsilon_{\alpha\beta}} \cdot \frac{\partial \varepsilon_{\alpha\beta}}{\partial v_{ij}}$$

and noting that as a result of (1.8) and (1.9)

$$\frac{\partial \varepsilon_{\alpha\beta}}{\partial v_{ij}} = \frac{1}{2}(\delta_{\alpha i}\delta_{\beta j} + \delta_{\beta i}\delta_{\alpha j} + \delta_{\alpha j} v_{i\beta} + v_{i\alpha}\delta_{\beta j}).$$

Here δ_{ij} denotes the Kronecker delta, i.e.

$$\left.\begin{aligned} \delta_{ij} = 1, \quad \text{if} \quad i = j, \\ \delta_{ij} = 0, \quad \text{if} \quad i \neq j. \end{aligned}\right\} \tag{1.12}$$

The equilibrium equations then assume the final form

$$\frac{\partial}{\partial x_j}\left[\frac{\partial \Phi}{\partial \varepsilon_{jk}}\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\right] + X_i = 0, \tag{1.13}$$

and the boundary conditions on the loaded surface S become

$$\frac{\partial \Phi}{\partial \varepsilon_{jk}}\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right) n_j = p_i. \tag{1.14}$$

1.4. GEOMETRICAL INTERPRETATION OF RESULTS

In order to establish the physical significance of the quantities $\partial \Phi / \partial \varepsilon_{jk}$ (they have the dimensions of stress and in the linear approximation become stresses in the normal sense) we proceed as follows:[1] we consider the deformed state of an infinitely small element which before deformation is a rectangular parallelepiped with edges dx_1, dx_2, dx_3 (Fig. 13). The internal forces acting on its faces will be represented by the vectors $\sigma_1\, dx_2\, dx_3$, $\sigma_2\, dx_1\, dx_3$, $\sigma_3\, dx_1\, dx_2$. We see that the components of vectors σ_1, σ_2, σ_3 are in fact stresses referred to areas of the element in its undeformed state.

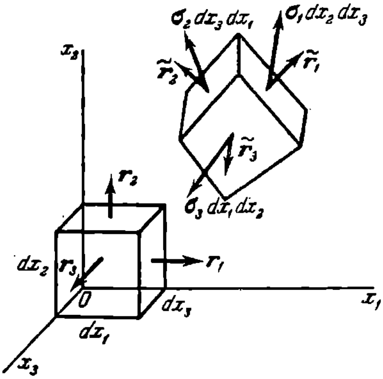


Fig. 13

The equilibrium equations and boundary conditions on the loaded surface, written in vector form, as is usual, are expressed by

$$\left.\begin{aligned} \frac{\partial \sigma_j}{\partial x_j} + \mathrm{X} &= 0, \\ \sigma_j n_j &= p. \end{aligned}\right\} \tag{1.15}$$

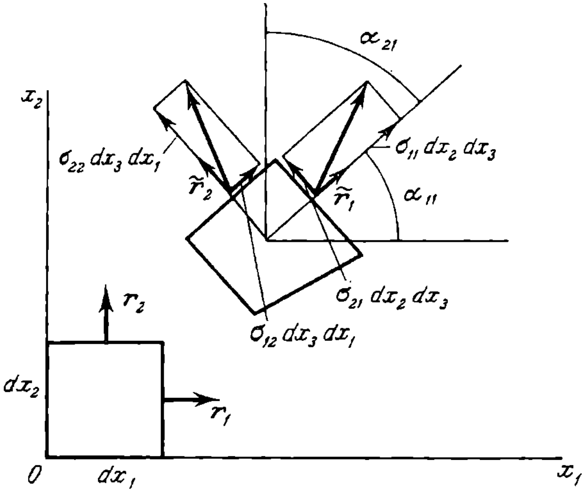
---

[1] KAPPUS, R. Zur Elastizitätstheorie endlicher Verschiebungen. *Z. angew. Math. Mech.* 19 (1939); NOVOZHILOV, V. V. See reference on p. 28.

We can rewrite these expressions in terms of the projections on the axes x_1, x_2, x_3, i.e. in terms of Lagrange variables for the undeformed body. Instead of (1.15) we then have

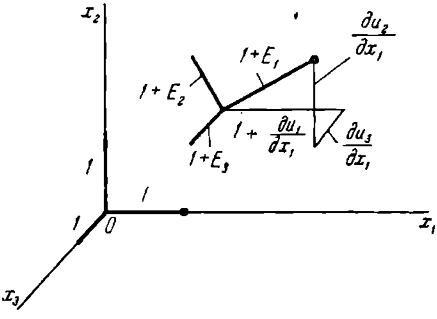
$$\left.\begin{aligned}\frac{\partial}{\partial x_j}(\sigma_{jk}\, l_{ik}) + X_i = 0,\\ \sigma_{jk}\, l_{ik}\, n_j = p_i,\end{aligned}\right\} \tag{1.16}$$

where X_i and p_i have their previous meanings, σ_{jk} are the projections of the vector σ_j on the coordinate directions for the deformed element, l_{ik} are the cosines of the angles between the directions of the axis x_i and the coordinate vector $\tilde{r}_k$ for the deformed element (in the plane problem $l_{ik} = \cos\alpha_{ik}$, see Fig. 14).



x_2
α_{21}
$\sigma_{22}\, dx_3\, dx_1$
$\sigma_{11}\, dx_2\, dx_3$
$\tilde{r}_2$
$\tilde{r}_1$
α_{11}
$\sigma_{21}\, dx_2\, dx_3$
r_2
$\sigma_{12}\, dx_3\, dx_1$
dx_2
r_1
0
dx_1
x_1

Fig. 14



x_2
$\frac{\partial u_2}{\partial x_1}$
$1+E_1$
$1+E_2$
$1+\frac{\partial u_1}{\partial x_1}$
$\frac{\partial u_3}{\partial x_1}$
$1+E_3$
1
1
1
0
x_1
x_3

Fig. 15

From geometrical considerations (Fig. 15) it follows that

$$l_{ik} = \frac{\delta_{ik} + \dfrac{\partial u_i}{\partial x_k}}{1 + E_k}, \tag{1.17}$$

where E_k is the relative extension of a fiber which in the undeformed state lay in the direction of the x_k axis and δ_{ik} is, as before, the Kronecker delta defined by (1.12). If we now substitute (1.17) into (1.16) we obtain

$$\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\delta_{ik} + \dfrac{\partial u_i}{\partial x_k}}{1 + E_k}\right) + X_i = 0,$$

$$\left(\sigma_{jk}\frac{\delta_{ik} + \dfrac{\partial u_i}{\partial x_k}}{1 + E_k}\right) n_j = p_i,$$

whence, by comparison with (1.13) and (1.14), we find

$$\frac{\partial \Phi}{\partial \varepsilon_{jk}} = \frac{\sigma_{jk}}{1 + E_k}. \tag{1.18}$$

Formula (1.18) enables us to give the following interpretation of the tensor $\partial\Phi/\partial\varepsilon_{jk}$: to the accuracy of the approximation $1 + E_k \approx 1$ the quantities $\partial\Phi/\partial\varepsilon_{jk}$ are components of the stress tensor in Lagrange variables referred not to the actual areas, but to areas measured before deformation. As has already been pointed out, in questions of elastic stability it is admissible to consider displacements (including angles of rotation) as finite and to take changes in areas and linear dimensions as negligibly small. In this case the difference between $\partial\Phi/\partial\varepsilon_{jk}$ and σ_{jk} disappears; so also does the difference between the stresses σ_{jk}, referred to initial areas, and the "true" stresses in Lagrange variables. We shall use the equations

$$\frac{\partial}{\partial x_j}\left[\sigma_{jk}\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right)\right] + X_i = 0 \tag{1.19}$$

and the boundary conditions

$$\sigma_{jk}\left(\delta_{ik} + \frac{\partial u_i}{\partial x_k}\right) n_j = p_i \tag{1.20}$$

as the basis of our subsequent analysis. As has already been mentioned, the accuracy of equations (1.19) and (1.20) is not lower than the accuracy of the corresponding applied theories of bars, plates and shells.

1.5. Relation between Stresses and Strains

The properties of a material are characterized by the expression for the density of its internal energy

$$\Phi = \Phi(\varepsilon_{11}, \varepsilon_{12}, \dots, \varepsilon_{33}).$$

As in the classical theory of elasticity, it can be shown that

$$\Phi = \tfrac{1}{2} \lambda_{ik\alpha\beta} \varepsilon_{ik} \varepsilon_{\alpha\beta}; \tag{1.21}$$

here $\lambda_{ik\alpha\beta}$ is the tensor of the elastic constants, which is symmetrical with respect to exchange of indices i and k, α and β and also of pairs. For an isotropic body the tensor $\lambda_{ik\alpha\beta}$ can be expressed in terms of two Lamé parameters λ and μ:

$$\lambda_{ik\alpha\beta} = \lambda\, \delta_{ik}\, \delta_{\alpha\beta} + 2\mu\, \delta_{i\alpha}\, \delta_{k\beta}. \tag{1.22}$$

Making use of the approximate relation

$$\sigma_{ik} \approx \frac{\partial\Phi}{\partial\varepsilon_{ik}},$$

we obtain from (1.21) the usual relation between the stresses and strains

$$\sigma_{ik} = \lambda_{ik\alpha\beta}\varepsilon_{\alpha\beta}. \tag{1.23}$$

1.6. Curvilinear Coordinates

The foregoing theory was based on the very simple case of a system of rectangular Cartesian coordinates. In many branches of the theory of elasticity, however, and especially in the theory of elastic shells, the use of curvilinear coordinates becomes absolutely unavoidable.

In order to present the equations of the non-linear theory of elasticity in a form suitable for any system of curvilinear coordinates, we employ the methods of tensor analysis.[1] In essence, the basic problem of tensor analysis is the investigation of quantities and mathematical operations on them which retain one and the same form for as large a class of coordinate systems as possible. The application of tensor analysis considerably simplifies the transformation of equations in a system of rectangular Cartesian coordinates to corresponding equations in a system of curvilinear coordinates, reducing this process to a formal operation.

---

[1] Kochin, N. E. *Vector Analysis and the Fundamentals of Tensor Analysis* (Vektornoe ischislenie i nachala tensornogo ischisleniia). Akad. Nauk USSR, 1951. An equivalent reference to a text in English would be: Sokolnikoff, I. S. *Tensor Analysis*, New York 1951.

The position of a point in space can be defined by three curvilinear coordinates q_1, q_2, q_3. We shall assume that the functions

$$q_i = q_i(x_1, x_2, x_3) \tag{1.24}$$

are single-valued, continuous and have continuous derivatives of any order required. We shall also assume that the Jacobian of the transformation (1.24) is everywhere non-zero.

An important property of a system of curvilinear coordinates is its metric, defined by the square of the distance between two infinitely close points in space. In a rectangular system of coordinates this distance is given by the formula

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2,$$

and in the general case of curvilinear coordinates by

$$ds^2 = g_{ik}\, dq_i\, dq_k. \tag{1.25}$$

Here g_{ik} is the so-called metric tensor, and $g_{ik} = g_{ki}$. In the case of a Cartesian system of coordinates, $g_{ik} = \delta_{ik}$, where δ_{ik} is the Kronecker delta. For cylindrical coordinates r, φ, z we have:

$$ds^2 = dr^2 + r^2\, d\varphi^2 + dz^2,$$

i.e. the metric tensor is given by

$$g_{ik} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

Let us now consider the ways of defining a vector in an arbitrary system of coordinates. A vector A will be defined if its orthogonal projections on the directions of the tangents to the coordinate axes are given. In elementary mechanics it is these components that are used, and in tensor analysis they are called *physical components*. They prove unsuitable, however, for use in the derivation of a general theory. They are, in fact, the components of a vector only for a local

Cartesian system of coordinates (in general, an oblique system) i.e., they are not the "true" components for a curvilinear coordinate system.

This can be illustrated by considering the case of cylindrical coordinates r, φ, z. The physical components of the differential of the radius vector are dr, $r\,d\varphi$, dz, but it would be simpler formally, and from the mathematical point of view more natural, to define them as dr, $d\varphi$, dz. Their physical dimensions now become different, although they assume a common dimension in a wider sense—a common tensorial dimension. Similarly for an arbitrary vector A; in addition to the physical components A_r, A_φ, A_z, we can introduce the components

$$a^r = A_r, \qquad a^\varphi = \frac{A_\varphi}{r}, \qquad a^z = A_z.$$

Their dimensions coincide with those of dr, $d\varphi$, dz, and furthermore, in transferring from one coordinate system to another, they obey the same transformation law as the differentials of coordinates dr, $d\varphi$, dz. They are called *contravariant* components.

It is customary to denote the contravariant components by superscripts (which of course, must not be confused with their power sign). Since the differentials of the coordinates are contravariant vectors, the coordinates themselves are usually given superscripts: $q_1 = x^1$, $q_2 = x^2$, $q_3 = x^3$. In transferring from one coordinate system to another, i.e. in the transformation

$$\bar{x}^i = \bar{x}^i(x^1, x^2, x^3),$$

the contravariant components are transformed according to the formulae

$$\bar{a}^i = \frac{\partial \bar{x}^i}{\partial x^k} a^k.$$

In addition to contravariant components of a vector, we also have the *covariant* components $a_i = g_{ik}\, a^k$. For instance, in the preceding case of a cylindrical system of coordinates the covariant components of a vector would be $a_r = A_r$, $a_\varphi = r A_\varphi$, $a_z = A_z$. The transformation law for covariant components is of the form

$$\bar{a}_i = \frac{\partial x^k}{\partial \bar{x}^i} a_k.$$

The relation between these components and the physical components is given by the formulae

$$A_i = \frac{a_i}{\sqrt{g_{ii}}} \tag{1.26}$$

(here summation is not carried out with respect to the subscript i!).

For a rectangular Cartesian system of coordinates $g_{ik} = \delta_{ik}$, and consequently $a_k = a^k = A^k$. In this case there is no difference between these components.

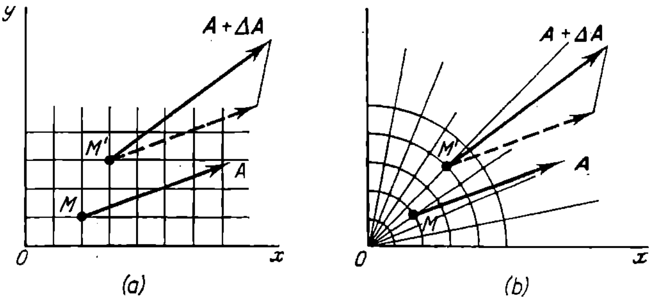
A vector can be looked upon as a tensor of the first order (and a scalar as a zero-order tensor). A second order tensor can be defined with the aid of covariant components b_{ik}, contravariant components b^{ik} and mixed components b^i_k or b^k_i (if the tensor is symmetrical its mixed components can be denoted simply by b^i_k). In passing from one coordinate system to another the components are transformed according to certain rules which, strictly speaking, should indicate that the given numbers characterize a second order tensor. For example the transformation law for covariant components is of the form

$$\bar{b}_{ij} = \frac{\partial x^\alpha}{\partial \bar{x}^i} \frac{\partial x^\beta}{\partial \bar{x}^j} b_{\alpha\beta}.$$

The passage from one set of components to another leads to the operations of "raising" or "lowering" the index by multiplying by the metric tensor. For example

$$b_{ij} = g_{i\alpha} g_{j\beta} b^{\alpha\beta}. \tag{1.26b}$$

The mixed components of a metric tensor are in fact the Kronecker delta, i.e. $g^i_j = \delta_{ij}$.



(a) (b)

Fig. 16

We proceed now to a most important operation of tensor analysis—the operation of *tensor differentiation.* Let us first consider the very simple case of a system of rectangular Cartesian coordinates and a system of polar coordinates in one plane (Fig. 16). The derivative of a vector is represented by the change in the vector in going from one point in the plane to another infinitely close point. The question arises as to how this change can be described in terms of the derivatives of the components of the vector.

In the case of the rectangular Cartesian coordinates the partial derivatives

$$\frac{\partial A_x}{\partial x}, \quad \frac{\partial A_y}{\partial y}, \quad \frac{\partial A_x}{\partial y}, \quad \frac{\partial A_y}{\partial x}$$

form a tensor—the first derivative of the vector A, also known as the divergence of the vector. In the case of the polar coordinates the partial derivatives

$$\frac{\partial A_r}{\partial r}, \quad \frac{\partial A_\varphi}{\partial \varphi}, \quad \frac{\partial A_r}{\partial \varphi}, \quad \frac{\partial A_\varphi}{\partial r},$$

as they stand cannot describe the change in the vector A. This is explained by the fact that if the vector A is displaced from the point M to a neighboring point M' (Fig. 16) with its direction unaltered, the space metric changes, and this change must be taken into account in evaluating the derivatives of the vector. The reader will be well aware of this fact without any reference to tensor analysis. For example, in the classical theory of elasticity the relative extensions are expressed in terms of the displacements u and v in the directions of the coordinate axes in different ways for different coordinate systems: whereas in a Cartesian system

$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{yy} = \frac{\partial v}{\partial y},$$

in a system of polar coordinates we have

$$e_{rr} = \frac{\partial u}{\partial r}, \qquad e_{\varphi\varphi} = \frac{1}{r}\frac{\partial v}{\partial \varphi} + \frac{u}{r},$$

and the second term in the formula for $e_{\varphi\varphi}$ takes into account the change in the metric (the formulae are written in "physical" components).

A tensorial derivative of a vector is equal to the corresponding partial derivative plus certain terms which take into account the change in the metric in passing from one point to another infinitely close point. Thus the covariant derivative of the covariant components a_i is given by the formula

$$\nabla_k a_i = \frac{\partial a_i}{\partial x^k} - \Gamma^\lambda_{ik} a_\lambda,$$

where Γ^λ_{ik} are the so-called *Christoffel symbols*, which are expressed in terms of the components of the metric tensor in the following manner:

$$\Gamma^\lambda_{ik} = g^{\lambda\mu}\Gamma_{\mu,ik}, \qquad \Gamma_{\mu,ik} = \frac{1}{2}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^\mu}\right).$$

The derivatives $\nabla_k a_i$ form a second order covariant tensor, but the Christoffel symbols (just as the partial derivatives of a vector) are not tensors.

Similarly, the covariant derivative of the contravariant components of a vector is given by the formula

$$\nabla_k a^i = \frac{\partial a^i}{\partial x^k} + \Gamma^i_{\lambda k} a_\lambda.$$

In some cases it is convenient to make use of the contravariant derivatives

$$\nabla^k a_i = g^{k\lambda}\nabla_\lambda a_i; \qquad \nabla^k a^i = g^{k\lambda}\nabla_\lambda a^i.$$

The rules for differentiating a second order tensor may be stated as:

$$\nabla_k b_{ij} = \frac{\partial b_{ij}}{\partial x^k} - \Gamma^\lambda_{ik} b_{\lambda j} - \Gamma^\lambda_{jk} b_{i\lambda},$$

$$\nabla_k b^{ij} = \frac{\partial b^{ij}}{\partial x^k} + \Gamma^i_{k\lambda} b^{\lambda j} + \Gamma^j_{\lambda k} b^{i\lambda}$$

(there is no need to state the analogous rules for the contravariant derivatives and the derivatives of the mixed components).

1.7. EQUATIONS OF THE NON-LINEAR THEORY OF ELASTICITY IN AN ARBITRARY SYSTEM OF CURVILINEAR COORDINATES

If an equation contains terms of a tensorial character, then the passage from this equation written for a rectangular Cartesian system of coordinates to an equation suitable for any system of curvilinear coordinates can be carried out very simply. The partial derivatives are replaced by the corresponding tensorial derivatives, and the "physical" components of vectors and tensors by their covariant and contravariant components. The type of components are chosen in such a way that the requirements of tensorial dimensionality are satisfied, i.e. so that each term appearing in the equation has the same tensorial order, and the "free" indices, with respect to which the summation is not carried out, are the same in all terms. For example, in place of the formulae for the components of the strain tensor in Cartesian coordinates (1.4), we obtain a formula for the covariant components in an arbitrary system of curvilinear coordinates:

$$\varepsilon_{ik} = \tfrac{1}{2}(\nabla_k u_i + \nabla_i u_k + \nabla_i u_j \nabla_k u^j). \tag{1.27}$$

The equilibrium equations (1.13) and the boundary conditions (1.14) can be written in the form

$$\nabla_j [\sigma^{jk}(g_k^i + \nabla_k u^i)] + X^i = 0, \tag{1.28}$$

$$\sigma^{jk}(g_k^i + \nabla_k u^i)\, n_j = p^i. \tag{1.29}$$

Here X^i and p^i are the contravariant components of the vector of the body forces and of the vector of surface loading,

$$\sigma^{jk} = \frac{\partial \Phi}{\partial \varepsilon_{jk}},$$

where, as before, Φ is the density of the elastic energy of the body (a scalar).

All the terms in Formula (1.27) have the dimensions of a covariant tensor of the second order with free indices i and k; all the terms in Formulae (1.28) and (1.29) have the dimensions of a contravariant vector with a free index i. Wherever the summation is carried out with respect to "dummy" indices, these indices occur twice: once as a superscript and the second time as a subscript. If we multiply each of the relations by the components of the metric tensor, we obtain the same physical relations, but they now have different tensorial dimensions.

In order to transfer to some particular system of curvilinear coordinates, it is necessary to evaluate the Christoffel symbol for this system and express the tensorial derivatives in terms of the usual partial derivatives. In addition, if it is expedient to do so, we can transfer to the "physical" components by means of formulae of the type (1.26). In the future we shall resort to curvilinear coordinates only when it is essential to do so.

1.8. Formulation of the Stability Problem

Variational Equations

We shall proceed now to the problem of elastic stability. The concept of "stability" or "instability" is in itself difficult to define, and is subject to a number of different interpretations, a fact which more than once has led to misunderstandings.[1] A significant step forward was the definition of elastic stability (or, in general, of motion) suggested by Liapunov, which, as well as being mathematically exact, is in accordance with our intuitive ideas of this phenomenon based on everyday experience, and corresponds to the majority of engineering applications. Liapunov's definition, expressed in mathematical terms, can be found in any textbook on the theory of stability of motion or in a sufficiently comprehensive course on the theory of oscillations.[2]

In the future we shall always speak of the stability (or instability) of a certain form of *equilibrium*. This will be called the *undisturbed* form of equilibrium. In addition, we shall also consider *disturbed* forms of motion, close to the undisturbed form of equilibrium.

We shall define *stable* equilibrium as a state in which small disturbances lead to small deviations from the equilibrium position, and in which, however small the disturbances are made, the deviations from the position of undisturbed equilibrium are still reduced by corresponding amounts. If the deviations from the equilibrium position, which are small, tend asymptotically to zero as the time $t \to \infty$, then the equilibrium is known as asymptotically stable. This, obviously, is an even more definite case of stability than the previous definition. Finally, if a disturbance, no matter how small, causes a finite deviation from the undisturbed state of equilibrium, then the latter is unstable.

These definitions are not, of course, precise. There still remains the question of what exactly is meant by a "disturbance" which causes a deviation from the undisturbed state of equilibrium, and how the deviation itself should be defined. If we are dealing with the stability of some elastic body (for instance, a bar, a framework, a plate or a shell) then the disturbance can be produced by various slight impulses, small increases in load not taken into account in the analysis, small defects and deviations from the specified dimensions, etc. The measure of deviation from the undisturbed state of equilibrium can be taken both in terms of the displacements and strains, and of the additional stresses and other characteristics of force.

---

[1] Bellman has defined the concept of stability as a "heavily loaded term with an unstable definition" (Bellman, R. E. *Stability Theory of Differential Equations*, McGraw-Hill, N. Y. 1953.

[2] See, for example, Chetaev, N. G. *Stability of Motion* (Ustoichivost' dvizheniia). Gostekhizdat, 1956.

In many cases an assessment of stability (or instability) of equilibrium can be made by assuming that the disturbances are sufficiently small, and carrying out the investigation on the basis of linearized differential equations. As suggested by Poincaré, these are called *variational equations.* We will show how they can be applied to the problem of the stability of equilibrium of an elastic body.

Let us consider an elastic body in a state of undisturbed equilibrium defined by a displacement vector u_i, a stress tensor σ_{ik} and vectors of body and surface forces X_i and p_i. The characteristics of the undisturbed equilibrium must satisfy Equations (1.19) and the boundary conditions on the surface (1.20):

$$\left.\begin{aligned} \frac{\partial}{\partial x_j}\left[\sigma_{jk}\left(\delta_{ik}+\frac{\partial u_i}{\partial x_k}\right)\right]+X_i&=0,\\ \sigma_{jk}\left(\delta_{ik}+\frac{\partial u_i}{\partial x_k}\right)n_j&=p_i. \end{aligned}\right\} \tag{1.30}$$

We cause the body to assume certain small deviations from the position of undisturbed equilibrium and investigate the change in these disturbances with time. The components of the characteristics of the disturbed motion (we shall distinguish them by the sign ~, and the disturbances by a bar) will be of the form

$$\left.\begin{aligned} \tilde{u}_i&=u_i+\bar{u}_i,\\ \tilde{\sigma}_{ik}&=\sigma_{ik}+\bar{\sigma}_{ik},\\ \tilde{X}_i&=X_i+\bar{X}_i-\varrho\frac{\partial^2\bar{u}_i}{\partial t^2},\\ \tilde{p}_i&=p_i+\bar{p}_i. \end{aligned}\right\} \tag{1.31}$$

Here $\bar{X}_i$ and $\bar{p}_i$ are the disturbances of the body and surface forces, which in general depend on t. Strictly speaking, inertia forces are also disturbances of body forces, but for convenience they will not be included; ϱ is the density of the material of the body.

The components of disturbed motion must also satisfy equations and boundary conditions of the type (1.30):

$$\frac{\partial}{\partial x_j}\left[\tilde{\sigma}_{jk}\left(\delta_{ik}+\frac{\partial \tilde{u}_i}{\partial x_k}\right)\right]+\tilde{X}_i=0,$$

$$\tilde{\sigma}_{jk}\left(\delta_{ik}+\frac{\partial \tilde{u}_i}{\partial x_k}\right)n_j=\bar{p}_i.$$

Substituting (1.31) into these, and taking into account (1.30), we obtain the equation of disturbed motion

$$\frac{\partial}{\partial x_j}\left[\bar{\sigma}_{jk}\left(\delta_{ik}+\frac{\partial u_i}{\partial x_k}+\frac{\partial \bar{u}_i}{\partial x_k}\right)\right]+\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right)\bar{X}_i-\varrho\frac{\partial^2 \bar{u}_i}{\partial t^2}=0 \tag{1.32}$$

and the boundary conditions

$$\bar{\sigma}_{jk}\left(\delta_{ik}+\frac{\partial u_i}{\partial x_k}+\frac{\partial \bar{u}_i}{\partial x_k}\right)n_j+\sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k}n_j=\bar{p}_i. \tag{1.33}$$

We now make use of the condition that the disturbances are small. This enables us to discard the terms $\bar{\sigma}_{jk}\cdot\partial\bar{u}_i/\partial x_k$ from the equations, since they start with terms of the second order of smallness. The tensor $\bar{\sigma}_{ik}$ can easily be expressed in terms of the vector $\bar{u}_i$, by making use of Formulae (1.4) and (1.23). To the accuracy of terms linear with respect to $\bar{u}_i$, we have

$$\bar{\sigma}_{ik}=\frac{1}{2}\lambda_{ik\alpha\beta}\left(\frac{\partial \bar{u}_\alpha}{\partial x_\beta}+\frac{\partial \bar{u}_\beta}{\partial x_\alpha}\right)=\lambda_{ik\alpha\beta}\frac{\partial \bar{u}_\alpha}{\partial x_\beta}.$$

(in which we have made use of the properties of symmetry of the tensor $\lambda_{ik\alpha\beta}$).

The final simplification, which we shall now introduce, is of a more special nature. In the vast majority of engineering problems the difference between the undisturbed state and the initial undeformed state is comparatively small, and it is only when loss of stability occurs that large increases in displacements occur. Therefore, in the solution of practical problems of elastic stability the geometry in the undisturbed state of equilibrium, the stability of which is the subject of investigation, is usually taken to be the same as that in the undeformed state. The number of problems in which the influence of displacements preceding loss of stability has been taken into account is extremely small. As an example, we can cite the problem investigated by

Pflüger[1] of the stability of the plane form of bending of a strip, taking into account an initial deflection in the plane of maximum stiffness.

The influence of displacements which precede loss of stability is taken into account in Equations (1.32) and in the boundary conditions (1.33) by the terms containing $\partial u_i/\partial x_k$. If we omit these terms we effect a further simplification of the equations. Furthermore, it should be noted that the investigation of the effect of these terms is in itself an independent branch of the theory of elastic stability.

After all these simplifications the variational equations and the corresponding boundary conditions now become

$$\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right) + \bar{X}_i - \varrho \frac{\partial^2 \bar{u}_i}{\partial t^2} = 0, \tag{1.34}$$

$$\bar{\sigma}_{ij} n_j + \sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k} n_j = \bar{p}_i. \tag{1.35}$$

These must be considered together with the relations

$$\bar{\sigma}_{ij} = \lambda_{ij\alpha\beta}\frac{\partial \bar{u}_\alpha}{\partial x_\beta}. \tag{1.36}$$

The undisturbed equilibrium will be stable or unstable depending on the behavior of the disturbances as $t \to \infty$. If all the solutions to the system (1.34) with boundary conditions (1.35) decrease with time, then the equilibrium is stable. The frequently encountered case, when the system (1.34) with boundary conditions (1.35) has only a periodic solution, is by convention also classified as stable, (stability in the sense of Kelvin). If even one of the solutions is found to increase infinitely with time, the equilibrium is unstable. We assume here that theorems analogous to the classical theorems of the theory of stability of motion can be proved, which allow us to assess the stability of the non-linear continuous system by means of the linearized equations of motion.

It is assumed that the forces acting on the system are given to the accuracy of some parameter β. The problem is to find the critical value of β at which the undisturbed form of equilibrium ceases to be stable.

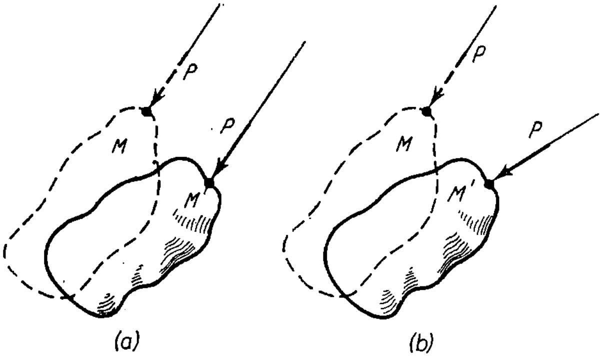
---

[1] PFLÜGER, A. *Stabilitätsprobleme der Elastostatik*, Springer-Verlag 1950.

1.9. VARIOUS CASES OF LOAD BEHAVIOR

It is well known that the behavior of the load when the body deviates slightly from its position of undisturbed equilibrium, can have a significant effect on its stability (or instability). We shall now investigate this question in a little more detail.

The most common type of load is a "dead" load, which remains constant both in magnitude and in direction, and which for small displacements of the body from its initial position is displaced together with the corresponding point on the body (Fig. 17a). In their projections onto fixed axes the components of such a load obviously remain constant, i.e. $\bar{X}_i = \bar{p}_i = 0$.



(a) (b)

Fig. 17

Let us now consider the case of a "follower" force which, whilst remaining constant in magnitude and moving together with the body, rotates in such a way that the angles formed by the vector of the load with the coordinate vectors on a Lagrangian basis remain constant (Fig. 17b).

The direction cosines of the coordinate vectors after deformation are given by Formula (1.17):

$$l_{ij} = \frac{\delta_{ij} + \dfrac{\partial \bar{u}_i}{\partial x_j}}{1 + \bar{E}_j} \approx \delta_{ij} + \frac{\partial \bar{u}_i}{\partial x_j},$$

whence the disturbed values of the load are

$$\left.\begin{aligned} \tilde{X}_i &= X_j l_{ij} \approx X_i + X_j \frac{\partial \bar{u}_i}{\partial x_j}, \\ \tilde{p}_i &= p_j l_{ij} \approx p_i + p_j \frac{\partial \bar{u}_i}{\partial x_j}. \end{aligned}\right\} \tag{1.37}$$

Thus, in the case of a "follower" force, we have

$$\bar{X}_i = X_j \frac{\partial \bar{u}_i}{\partial x_j}, \qquad \bar{p}_i = p_j \frac{\partial \bar{u}_i}{\partial x_j}. \tag{1.38}$$

We shall assume that the body and surface forces form a field dependent upon the Euler coordinates of a point. In this case

$$\begin{aligned} \tilde{X}_i(x_1, x_2, x_3) &= X_i(x_1 + \bar{u}_1, x_2 + \bar{u}_2, x_3 + \bar{u}_3) = \\ &= X_i + \frac{\partial X_i}{\partial x_k} \bar{u}_k + \cdots, \\ \tilde{p}_i(x_1, x_2, x_3) &= p_i(x_1 + \bar{u}_1, x_2 + \bar{u}_2, x_3 + \bar{u}_3) = \\ &= p_i + \frac{\partial p_i}{\partial x_k} \bar{u}_k + \cdots, \end{aligned}$$

whence

$$\bar{X}_i = \frac{\partial X_i}{\partial x_k} \bar{u}_k, \qquad \bar{p}_i = \frac{\partial p_i}{\partial x_k} \bar{u}_k. \tag{1.39}$$

A more specialized example is the case of aerodynamic forces acting on a body placed in a flow of liquid or gas. These forces can be calculated most simply when the body is in a flow of non-viscous compressible gas travelling with velocity U, considerably in excess of the speed of sound c_∞ (Fig. 18). In this case the pressure of the gas on the disturbed surface is made up of an undisturbed pressure p_∞ and an excess pressure, which is given by the approximate formula[1]

$$p = \frac{\varkappa p_\infty}{c_\infty} \left(\frac{\partial \bar{u}_3}{\partial t} + U \frac{\partial \bar{u}_3}{\partial x_1} \right). \tag{1.40}$$

Here $\varkappa$ is the ratio of specific heats of the gas. Thus the disturbance vector of the surface load for a flat plate will be

$$\bar{p}_i = \left\| \begin{matrix} 0 \\ 0 \\ \dfrac{\varkappa p_\infty}{c_\infty} \left(\dfrac{\partial \bar{u}_3}{\partial t} + U \dfrac{\partial \bar{u}_3}{\partial x_1} \right) \end{matrix} \right\|.$$

---

[1] For details see Chapter 4.

The cases considered above can be combined to form one general relation of the type

$$\left.\begin{aligned} \bar{X}_i &= a_{ij}\bar{u}_j + b_{ijk}\frac{\partial \bar{u}_j}{\partial x_k} + c_{ij}\frac{\partial \bar{u}_j}{\partial t} + d_{ijk}\frac{\partial^2 \bar{u}_j}{\partial x_k\,\partial t} + \cdots, \\ \bar{p}_i &= a'_{ij}\bar{u}_j + b'_{ijk}\frac{\partial \bar{u}_j}{\partial x_k} + c'_{ij}\frac{\partial \bar{u}_j}{\partial t} + d'_{ijk}\frac{\partial^2 \bar{u}_j}{\partial x_k\,\partial t} + \cdots \end{aligned}\right\} \tag{1.41}$$

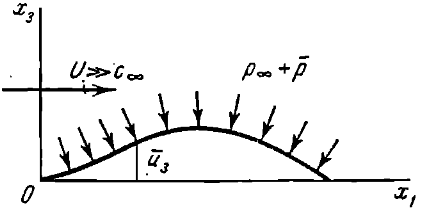


Fig. 18

Here a_{ij}, b_{ijk}, . . ., d'_{ijk} are coefficients which are independent of the vector $\bar{u}_k$ and its derivatives, but which in general vary from point to point.

1.10. STATIC BOUNDARY-VALUE PROBLEM

Let us consider the static problem corresponding to the system of equations (1.34) with the boundary conditions (1.35). We shall assume that all the external forces are given to an accuracy of a parameter β. We can then write

$$\sigma_{jk} = \beta\,\sigma^*_{jk}, \qquad \bar{X}_i = \beta\,\bar{X}^*_i, \qquad \bar{p}_i = \beta\,\bar{p}^*_i,$$

where σ^*_{jk}, $\bar{X}^*_i$ and $\bar{p}^*_i$ are independent of β but are linearly dependent on $\bar{u}_i$. In this way we arrive at a homogeneous linear system of equations

$$\frac{\partial}{\partial x_j}\left(\lambda_{ij\alpha\beta}\frac{\partial \bar{u}_\alpha}{\partial x_\beta}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma^*_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right) + \beta\,\bar{X}^*_i = 0 \tag{1.42}$$

with the following boundary conditions on the surface S:

$$\lambda_{ij\alpha\beta}\frac{\partial \bar{u}_\alpha}{\partial x_\beta}n_j + \beta\,\sigma^*_{jk}\frac{\partial \bar{u}_i}{\partial x_k}n_j = \beta\,\bar{p}^*_i. \tag{1.43}$$

The asterisks in σ_{jk}^{*}, $\bar{X}_i^{*}$ and $\bar{p}_i^{*}$ will be discarded from now on. The classical problem of elastic stability amounts to finding the minimum real eigenvalues of a boundary-value problem. The question immediately arises as to the conditions to be imposed on the external loading so that the problem presented in this way has a solution. A general answer to this question does not exist. It is, however, not difficult to obtain sufficient conditions so that the boundary-value problem has only real eigenvalues.

It is wellknown[1] that *if a linear boundary-value problem is self-adjoint, all its eigenvalues are real.* Suppose that $\bar{u}_i$ and $\bar{v}_i$ are two vectors with twice-differentiable components, which statisfy the boundary conditions (1.43) on the surface S and the conditions $\bar{u}_i = 0$, $\bar{v}_i = 0$ on the remainder of the surface of the body. The boundary-value problem is *self-adjoint* if, by virtue of the boundary conditions, the expression

$$\Delta = \int\limits_V \bar{v}_i\, \mathfrak{L}_i(\bar{u}_1, \bar{u}_2, \bar{u}_3)\, dV - \int\limits_V \bar{u}_i\, \mathfrak{L}_i(\bar{v}_1, \bar{v}_2, \bar{v}_3)\, dV = 0 \qquad (1.44)$$

vanishes for any choice of the vectors $\bar{u}_i$ and $\bar{v}_i$ satisfying these conditions. Here $\mathfrak{L}_i$ are the left-hand sides of Equations (1.42).

We can prove that Condition (1.44) is satisfied when $\bar{X}_i = \bar{p}_i = 0$. Applying the Gauss–Ostrogradskii formula, we find that

$$\int\limits_V \bar{v}_i \frac{\partial}{\partial x_j}\left(\lambda_{ij\alpha\beta}\frac{\partial \bar{u}_\alpha}{\partial x_\beta}\right) dV - \int\limits_V \bar{u}_i \frac{\partial}{\partial x_j}\left(\lambda_{ij\alpha\beta}\frac{\partial \bar{v}_\alpha}{\partial x_\beta}\right) dV =$$

$$= \iint\limits_S \bar{v}_i\, \lambda_{ij\alpha\beta}\frac{\partial \bar{u}_\alpha}{\partial x_\beta} n_j\, dS - \iint\limits_S \bar{u}_i\, \lambda_{ij\alpha\beta}\frac{\partial \bar{v}_\alpha}{\partial x_\beta} n_j\, dS,$$

$$\int\limits_V \bar{v}_i \frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right) dV - \int\limits_V \bar{u}_i \frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial \bar{v}_i}{\partial x_k}\right) dV =$$

$$= \iint\limits_S \bar{v}_i\, \sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k} n_j\, dS - \iint\limits_S \bar{u}_i\, \sigma_{jk}\frac{\partial \bar{v}_i}{\partial x_k} n_j\, dS.$$

---

[1] See, for example, NAIMARK, M. A. *Linear Differential Operators* (Lineinye differentsial'nye operatory) Gostekhizdat, 1954; MIKHLIN, S. G. *The Problem of the Minimum of a Quadratic Functional* (Problema minimuma kvadratichnogo funktsionala) Gostekhizdat 1952, etc.

From this we obtain

$$\int_V \bar{v}_i\, \mathfrak{L}_i(\bar{u}_1, \bar{u}_2, \bar{u}_3)\, dV - \int_V \bar{u}_i\, \mathfrak{L}_i(\bar{v}_1, \bar{v}_2, \bar{v}_3)\, dV =$$

$$= \iint_S \bar{v}_i \left(\lambda_{ij\alpha\beta} \frac{\partial \bar{u}_\alpha}{\partial x_\beta} + \beta\, \sigma_{jk} \frac{\partial \bar{u}_i}{\partial x_k}\right) n_j\, dS -$$

$$- \iint_S \bar{u}_i \left(\lambda_{ij\alpha\beta} \frac{\partial \bar{v}_\alpha}{\partial x_\beta} + \beta\, \sigma_{jk} \frac{\partial \bar{v}_i}{\partial x_k}\right) n_j\, dS.$$

But the integrands on the right-hand side vanish in consequence of Conditions (1.43), from which it follows that when $\bar{X}_i = \bar{p}_i = 0$ ("dead" loading) the static boundary-value problem is self-adjoint. This is not an unexpected result. It can easily be shown that Equations (1.42) and the boundary conditions (1.43) constitute the Ostrogradskii-Euler equations and the natural boundary conditions, respectively, for the variational problem of finding stationary values of the functional

$$I = \frac{1}{2} \int_V \lambda_{ij\alpha\beta} \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_\alpha}{\partial x_\beta} + \frac{\beta}{2} \int_V \sigma_{jk} \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_k}\, dV. \qquad (1.45)$$

The first term on the right-hand side represents the increment in strain energy due to the deviation of the system from the equilibrium position under investigation. The second term represents the increment in the work done by the external forces, expressed in terms of the components of the tensor σ_{jk}. It will be noted that the functional I, to the accuracy of a factor $^1/_2$ is equal to the second variation $\Delta^2 Э$ of the total energy of the system $Э$. (This variation is evaluated on the assumption that the variations of the displacements are equal to the disturbances $\bar{u}_j$.)

Let us assume now that $\bar{X}_i \neq 0$, $\bar{p}_i \neq 0$. In this case the left-hand side of Condition (1.44) becomes

$$\Delta = \int_V \bar{v}_i\, \mathfrak{L}_i(\bar{u}_1, \bar{u}_2, \bar{u}_3)\, dV - \int_V \bar{u}_i\, \mathfrak{L}_i(\bar{v}_1, \bar{v}_2, \bar{v}_3)\, dV =$$

$$= - \left[\int_V \bar{v}_i\, \bar{X}_i(\bar{u}_1, \bar{u}_2, \bar{u}_3)\, dV + \iint_S \bar{v}_i\, \bar{p}_i(\bar{u}_1, \bar{u}_2, \bar{u}_3)\, dS\right] +$$

$$+ \left[\int_V \bar{u}_i\, \bar{X}_i(\bar{v}_1, \bar{v}_2, \bar{v}_3)\, dV + \iint_S \bar{u}_i\, \bar{p}_i(\bar{v}_1, \bar{v}_2, \bar{v}_3)\, dS\right].$$

In order that the boundary-value problem is self-adjoint, it is necessary and sufficient for the condition

$$\int\limits_V \bar{v}_i \bar{X}_i(\bar{u}_1, \bar{u}_2, \bar{u}_3)\, dV + \iint\limits_S \bar{v}_i \bar{p}_i(\bar{u}_1, \bar{u}_2, \bar{u}_3)\, dS =$$

$$= \int\limits_V \bar{u}_i \bar{X}_i(\bar{v}_1, \bar{v}_2, \bar{v}_3)\, dV + \iint\limits_S \bar{u}_i \bar{p}_i(\bar{v}_1, \bar{v}_2, \bar{v}_3)\, dS, \qquad (1.46)$$

to be satisfied. This condition has an obvious physical meaning: it states that the reciprocal work theorem is valid for the external load system applied to the body.

Condition (1.46) is equivalent to the condition that the external forces have a potential Π. If they do have a potential, then the virtual work done by the external forces in displacing the system from a state defined by a vector field $\bar{u}_j$ to a very close state $\bar{u}_j + \delta\, \bar{u}_j$ is a function of the state, and can be expressed in terms of the variation in the potential Π:

$$\int\limits_V (X_i + \bar{X}_i)\, \delta \bar{u}_i\, dV + \iint\limits_S (p_i + \bar{p}_i)\, \delta \bar{u}_i\, dS = \delta \Pi. \qquad (1.47)$$

The variation here is denoted by $\delta\Pi$, in contrast to $\Delta\Pi$, which refers to the change from the state u_j to the state $u_j + \bar{u}_j$. The equivalence of Conditions (1.46) and (1.47) can easily be proved in the same way as Betti's theorem in the theory of elasticity and structural mechanics, by basing the proof on the fact that the external and internal forces have a potential.

If the system of external forces has a potential, they are known as *conservative* forces. It should be noted that not all of the external forces need have a potential. It is only essential for the external load system applied to the body to have a potential, or in other words, for the force system *as a whole* to be conservative. Such a definition is in keeping with technical applications, in which the external forces can be defined as conservative or non-conservative, depending on whether or not external energy sources or sinks are present.

If Conditions (1.46) or (1.47) are satisfied, the stability problem in the Euler sense corresponds to the variational problem of the minimum of the functional

$$I = \frac{1}{2} \int\limits_V \lambda_{ij\alpha\beta} \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_\alpha}{\partial x_\beta}\, dV + \frac{\beta}{2} \int\limits_V \sigma_{jk} \frac{\partial \bar{u}_i}{\partial x_j} \frac{\partial \bar{u}_i}{\partial x_k}\, dV + \frac{1}{2} \Delta^2 \Pi.$$

Here $\Delta^2\Pi$ is the second variation of the external force potential. In the case of a "dead" load, a potential obviously exists and has the value

$$\Pi_0 = \int_V X_i \bar{u}_i \, dV + \iint_S p_i \bar{u}_i \, dS + \text{const.}$$

If the forces form a field which depends on Euler variables, then

$$\Pi = \Pi_0 + \frac{1}{2}\int_V \frac{\partial X_i}{\partial x_k} \bar{u}_i \bar{u}_k \, dV + \frac{1}{2}\iint_S \frac{\partial p_i}{\partial x_k} \bar{u}_i \bar{u}_k \, dS.$$

In the case of "follower" forces, a potential in general, does not exist. There are, however, certain exceptions. For instance, if

$$\int_V \bar{X}_i \, \delta\bar{u}_i \, dV + \iint_S \bar{p}_i \, \delta\bar{u}_i \, dS = 0$$

or, in other words, if

$$\int_V X_j \frac{\partial \bar{u}_i}{\partial x_j} \delta\bar{u}_i \, dV + \iint_S p_j \frac{\partial \bar{u}_i}{\partial x_j} \delta u_i \, dS = 0, \qquad (1.48)$$

the potential of the "follower" forces is Π_0. Condition (1.48) is satisfied, for example, if all the points of application of the external forces are fixed either against rotation or displacement. Then the other components of the "follower" forces, in which the latter differ from a "dead" load, do no work in displacements compatible with the constraints (Fig. 19).

The case of a load uniformly distributed over some part of the external surface of a body, and remaining normal to the deformed surface, is of particular interest. A load such as this is usually known as a hydrostatic load. If the load is in fact produced by the pressure of a liquid at rest, it is obviously a conservative force. However, loads are quite often represented in this way when they are not essentially conservative forces.[1]

Let us consider a shell referred to curvilinear coordinates x^1 and x^2 of its middle surface; x^3 is a coordinate measured along the external normal to the middle surface (Fig. 20). Suppose that dS is the area of an element of the middle surface before deformation. Then after deformation

---

[1] The remainder of this Section presupposes that the reader is familiar with elementary tensor analysis, at least to the level of the subject matter of Sections 1.6 and 1.7.

$$\widetilde{dS} = dS(1 + \nabla_\alpha u^\alpha),$$

where α takes the values 1 and 2. We shall assume that the shell is loaded by a uniform pressure p acting in a direction normal to the middle surface. After deformation the surface loading vector, to the accuracy of linear terms, will be given by

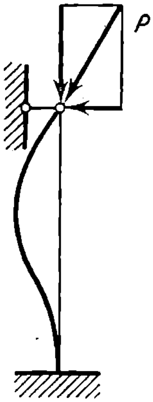


Fig. 19

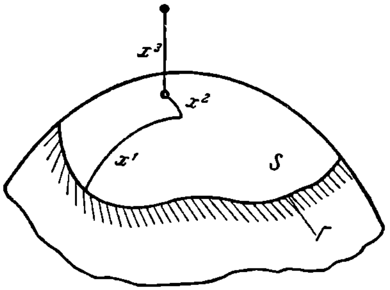


Fig. 20

$$\tilde{p}^i = \left\| \begin{matrix} -p\,\nabla_3 u^1 \\ -p\,\nabla_3 u^2 \\ -p \end{matrix} \right\|.$$

Hence the virtual work done by the external forces in the permissible displacements $\delta\, u_\alpha$ ($\alpha = 1, 2$), to the accuracy of linear terms, is

$$\delta A = -p \iint_S [(1 + \nabla_\alpha u^\alpha)\,\delta u_3 + \nabla_3 u^\alpha\, \delta u_\alpha]\, dS. \tag{1.49}$$

We now apply the Kirchhoff–Love hypothesis: $\nabla_3 u^\alpha = -\nabla^\alpha u_3$. Making use of the Gauss–Ostrogradskii formula

$$\iint_S \nabla^\alpha u_3\, \delta u_\alpha\, dS = \int_\Gamma u_3\, \delta u_\alpha\, d\Gamma^\alpha - \iint_S u_3\, \delta(\nabla_\alpha u^\alpha)\, dS$$

and assuming that the integral over the contour Γ bounding the loaded area S is zero, i.e.

$$\int_\Gamma u_3\, \delta u_\alpha\, d\Gamma^\alpha = 0, \tag{1.50}$$

we can write Expression (1.49) in the form

$$\delta A = -p \iint_S [(1 + \nabla_\alpha u^\alpha)\, \delta u_3 + u_3\, \delta(\nabla_\alpha u^\alpha)]\, dS.$$

It will readily be seen that the virtual work δA can now be expressed in terms of the potential

$$\Pi = -p \iint_S (1 + \nabla_\alpha u^\alpha)\, u_3\, dS.$$

For this to hold it is of course essential that the integral in (1.50) vanishes. It does so, for example, provided that everywhere on the boundary Γ either the normal displacement u_3, or the tangential displacement $u_\alpha d\Gamma^\alpha$, vanishes. If a part of a surface with a boundary which is free to move is under load, the so-called "hydrostatic" load is, in general, non-conservative. In this case it can not be produced, of course, merely by the pressure of a liquid at rest.[1]

1.11. OSCILLATIONS ABOUT THE EQUILIBRIUM POSITION AND EULER'S METHOD

Let us consider the equations

$$\frac{\partial}{\partial x_j}\left(\lambda_{ij\alpha\beta}\frac{\partial \bar{u}_\alpha}{\partial x_\beta}\right) + \beta \frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right) + \beta \bar{X}_i - \varrho \frac{\partial^2 \bar{u}_i}{\partial t^2} = 0, \qquad (1.51)$$

which describe small oscillations of a body about its equilibrium position, and the boundary conditions on the loaded surface S

$$\lambda_{ij\alpha\beta}\frac{\partial \bar{u}_\alpha}{\partial x_\beta} n_j + \beta\, \sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k} n_j = \beta\, \bar{p}_i. \qquad (1.52)$$

In the case of small oscillations we can take a solution in the form

$$\bar{u}_i(x_1, x_2, x_3, t) = U_i(x_1, x_2, x_3)\, e^{i\Omega t}, \qquad (1.53)$$

where U_i are components of a vector which defines the form of the oscillations, and Ω is the frequency, in general complex, of the oscillations. Substituting Expressions (1.53) into Equations (1.51) and boundary conditions (1.52); we obtain a new boundary-value problem:

---

[1] Some of the results of this Section were derived by other means in the paper: BOLOTIN, V. V. Some questions of the theory of elastic stability, *PMM*, Vol. **20**, No. 4 (1956).

$$\left.\begin{aligned}&\frac{\partial}{\partial x_j}\left(\lambda_{ij\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta}\right)+\beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial U_i}{\partial x_k}\right)+\\&\qquad\qquad+\beta\bar{X}_i(U_1,U_2,U_3,\Omega)-\varrho\,\Omega^2U_i=0,\\&\lambda_{ij\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta}n_j+\beta\,\sigma_{jk}\frac{\partial U_i}{\partial x_k}n_j=\beta\bar{p}_i(U_1,U_2,U_3,\Omega).\end{aligned}\right\}\quad(1.54)$$

The behavior of the eigenvalues Ω of this boundary-value problem, as the loading parameter β varies, is to be investigated. If all the eigenvalues $\Omega(\beta)$ have positive imaginary parts, or in the extreme case are real (we have already agreed to classify this as a case of stability, see Section 1.8), then the undisturbed motion is stable. If only one of the eigenvalues has a negative imaginary part, then, as will be seen from (1.53), this will indicate an increase in the disturbances with time, i.e. instability. If $\beta = 0$, all the eigenvalues of the boundary-value problem (1.54) are purely real (they are the natural frequencies of the unloaded elastic body). Let us suppose that the parameter β increases in modulus and remains positive (say). It is possible to find a value β_* an increase in which, no matter how small, causes one or more of the frequencies to enter the lower half-plane of the complex variable. This value corresponds to transition from stability to instability of the equilibrium, and is therefore the *critical value*. In the same way we can find a critical value in the region of negative β.

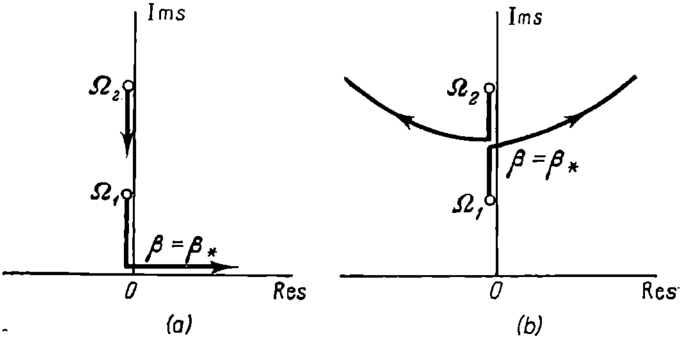


Fig. 21

Instead of the frequencies Ω, it is convenient to consider the *characteristic exponents* $s = i\Omega$. Provided all the exponents remain in the left-hand half-plane, the equilibrium under investigation is stable.

The smallest values in modulo of the parameter β, which correspond to even one exponent crossing into the right-hand half-plane, are critical values.

We must distinguish between two forms of transition from stability to instability. The first is when one exponent s enters the right-hand half-plane after passing through the origin of coordinates, and remains purely real at least for values of β sufficiently close to the critical (see Fig. 21a, in which only the upper half-plane is shown). The corresponding graph of the real parts of the frequencies Ω is shown in Fig. 22a. This form of instability is known as *static instability*. If the characteristic exponent s enters the right-hand half-plane at a point other than the origin of coordinates (Figs. 21 and 22), then this form of instability is known as *oscillatory instability*.[1] These two forms of instability differ in the behavior of the disturbances close to the boundary of stability: in the first case the disturbance increases monotonically with time, and in the second case the increase in the disturbance is of an oscillatory nature.

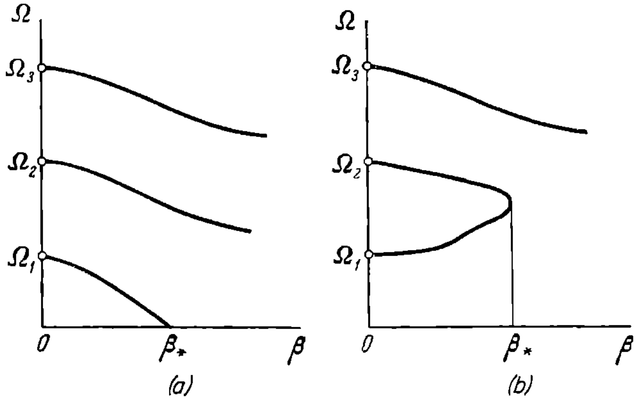


Fig. 22

Let us consider the boundary-value problem (1.54) on the assumption that the external forces have a potential. The expressions for $\bar{X}_i$ and $\bar{p}_i$ in Equations (1.54) are found to be independent of Ω, and we arrive at a boundary-value problem with eigenvalues $\Omega^2 = \Omega^2(\beta)$.

---

[1] It is true, of course, that static instability is also possible with transition through a point at infinity (cf. Fig. 5b).

Since for $\Omega = 0$ the boundary-value problem is self-adjoint, the boundary-value problem

$$\left.\begin{aligned} &\frac{\partial}{\partial x_j}\left(\lambda_{ij\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial U_i}{\partial x_k}\right) + \\ &\qquad\qquad + \beta\bar{X}_i(U_1, U_2, U_3) - \varrho\,\Omega^2 U_i = 0, \\ &\lambda_{ij\alpha\beta}\frac{\partial U_\alpha}{\partial x_\beta}n_j + \beta\,\sigma_{jk}\frac{\partial U_i}{\partial x_k}n_j = \beta\,\bar{p}_i(U_1, U_2, U_3) \end{aligned}\right\} \tag{1.55}$$

is also self-adjoint, and consequently all values of Ω^2 are real. Transition from stability to instability corresponds to a change from positive values of Ω^2 to negative values. Since the function $\Omega^2(\beta)$ is continuous, this transition must pass through the value of $\Omega^2 = 0$. Thus, *if the external forces have a potential, loss of stability can take place only in the form of static instability.*

In order to find the critical values of the parameter β, we set $\Omega^2 = 0$ in Equations (1.55). In this way we arrive at an eigenvalue problem (1.42), corresponding to the *Euler method* in the theory of elastic stability. If the external forces are conservative, the *Euler method* will give the correct solution to a stability problem without the need to resort to an investigation of small oscillations about the equilibrium position.

If the forces are non-conservative, the form which the loss of stability assumes requires special investigation in each problem. ***Both forms of instability are possible in this case.*** In a number of problems, depending on the relation between the parameters, the minimum critical loads can correspond to either the static or the oscillatory forms of loss of stability.

1.12. REDUCTION TO A SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

A direct analysis of the eigenvalues of the non-self-adjoint boundary-value problem (1.54) is extremely difficult, even in the simplest cases. The question arises, therefore, as to the possibility of using approximate methods of solution. The most effective approximate method is to express the components of the displacement vector in the form of a series in a system of coordinate functions which satisfy all or some of the boundary conditions. Adopting Galerkin's method, we can reduce the problem to one of finding a system of ordinary differential equations in the coefficients of a series, which are functions of time, and later to a purely algebraic problem.

In applying Galerkin's method to the non-self-adjoint problems, the question of convergence immediately arises. So far this question has been investigated only in connection with very simple self-adjoint problems. The wide use of Galerkin's method for problems in which convergence is not conclusively proved is based on a comparison of existing exact results with those obtained by Galerkin's method. It is quite certain, however, and it has been confirmed by experiment and exact calculations, that the forms of loss of stability with which we are concerned can be successfully represented to a good approximation by means of a linear combination of a small number of first modes of natural oscillations. At any rate, this has been the case in all problems investigated so far.[1]

Let us consider the system of equations

$$\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} + \beta \frac{\partial}{\partial x_j}\left(\sigma_{jk} \frac{\partial \bar{u}_i}{\partial x_k}\right) + \beta \bar{X}_i - \varrho \frac{\partial^2 \bar{u}_i}{\partial t^2} = 0 \qquad (1.56)$$

with the boundary conditions

$$\beta \bar{p}_i - \bar{\sigma}_{ij} n_j - \beta \sigma_{jk} \frac{\partial \bar{u}_i}{\partial x_k} n_j = 0. \qquad (1.57)$$

We shall try to find a solution to this system in the form of a series

$$\bar{u}_i(x_1, x_2, x_3, t) = \sum_{n=1}^{\infty} f_n(t)\, \varphi_{in}(x_1, x_2, x_3). \qquad (1.58)$$

Here $f_n(t)$ are certain functions of time, $\varphi_{in}(x_1, x_2, x_3)$ are coordinate vectors (three coordinate functions) necessarily satisfying all the geometrical boundary conditions but, in general, not satisfying the natural boundary Conditions (1.57). The system of functions $\varphi_{in}(x_1, x_2, x_3)$ is assumed to be complete.

The conditions imposed are satisfied, for example, by a system of the modes of natural oscillations of an unloaded elastic body. The equations of the natural oscillations

$$\frac{\partial}{\partial x_j}\left(\lambda_{ij\alpha\beta} \frac{\partial u_\alpha}{\partial x_\beta}\right) - \varrho \frac{\partial^2 u_i}{\partial t^2} = 0$$

---

[1] The reasons for the well-known difference between the exact and approximate solutions of the problem of membrane flutter in a supersonic flow are explained in Section 4.11.

with the boundary conditions on the surface S

$$\lambda_{ij\alpha\beta}\frac{\partial u_\alpha}{\partial x_\beta}n_j = 0$$

after making the substitution

$$u_i(x_1, x_2, x_3, t) = \varphi_i(x_1, x_2, x_3)\, e^{i\Omega t}$$

reduce to the equations

$$\frac{\partial}{\partial x_j}\left(\lambda_{ij\alpha\beta}\frac{\partial \varphi_\alpha}{\partial x_\beta}\right) + \varrho\,\Omega^2\,\varphi_i = 0 \tag{1.59}$$

with the boundary conditions

$$\lambda_{ij\alpha\beta}\frac{\partial \varphi_\alpha}{\partial x_\beta}n_j = 0. \tag{1.60}$$

All the natural frequencies Ω are real, and for a finite body form a discrete spectrum $\Omega_1 \leqq \Omega_2 \leqq \Omega_3 \leqq \cdots$ The corresponding eigenvectors $\varphi_{i1}(x_1, x_2, x_3)$, $\varphi_{i2}(x_1, x_2, x_3)$, $\varphi_{i3}(x_1, x_2, x_3)$, ... form a complete system of functions satisfying the condition of orthogonality

$$\iint\limits_V \varrho\,\varphi_{im}\,\varphi_{in}\,dV = \delta_{mn}. \tag{1.61}$$

Here δ_{mn} is the Kronecker delta, which is unity for $m = n$ and otherwise zero.

We apply the principle of virtual displacements to Equations (1.56) and boundary conditions (1.57). Introducing the virtual displacements $\delta\bar{u}_{i1}$, $\delta\bar{u}_{i2}$, $\delta\bar{u}_{i3}$, ..., which are compatible geometrically, and treating the left-hand sides of Equations (1.56) as components of the body forces, and the left-hand sides of (1.57) as components of the surface loads, we obtain:

$$\int\limits_V\left[\frac{\partial\bar{\sigma}_{ij}}{\partial x_j} + \beta\frac{\partial}{\partial x_j}\left(\frac{\partial\bar{u}_i}{\partial x_k}\right) - \beta\,X_i - \varrho\frac{\partial^2\bar{u}_i}{\partial t^2}\right]\delta\bar{u}_{im}\,dV + \\ + \iint\limits_S\left[\beta\bar{p}_i - \bar{\sigma}_{ij}\,n_j - \beta\,\sigma_{jk}\frac{\partial\bar{u}_i}{\partial x_k}n_j\right]\delta\bar{u}_{im}\,dS = 0 \tag{1.62}$$

$$(m = 1, 2, 3, \ldots).$$

We select the virtual displacements $\delta\bar{u}_{im}$ in the following way:

$$\delta\bar{u}_{im} = \delta f_m\,\varphi_{im}(x_1, x_2, x_3). \tag{1.63}$$

Substituting Expressions (1.63) into (1.62), and equating to zero the coefficients of arbitrary variations δf_m, we obtain the equations of the Galerkin method:

$$\int_V \left[\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} + \beta \frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right) - \beta \bar{X}_i - \varrho \frac{\partial^2 \bar{u}_i}{\partial t^2}\right] \varphi_{im}\, dV +$$

$$+ \iint_S \left[\beta \bar{p}_i - \bar{\sigma}_{ij} n_j - \beta\, \sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k} n_j\right] \varphi_{im}\, dS = 0 \qquad (1.64)$$

$$(m = 1, 2, 3, \ldots).$$

Equations (1.64) can be re-written if we take into account that

$$\int_V \frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right) \varphi_{im}\, dV - \iint_S \sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k} \varphi_{im}\, dS =$$

$$= -\int_V \sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\, \frac{\partial \varphi_{im}}{\partial x_j}\, dV.$$

Consequently,

$$\int_V \left[\frac{\partial \bar{\sigma}_{ij}}{\partial x_j} + \beta \bar{X}_i - \varrho \frac{\partial^2 \bar{u}_i}{\partial t^2}\right] \varphi_{im}\, dV - \beta \int_V \sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\, \frac{\partial \varphi_{im}}{\partial x_j}\, dV -$$

$$- \iint_S (\bar{\sigma}_{ij} n_j - \beta \bar{p}_i)\, \varphi_{im}\, dS = 0$$

$$(m = 1, 2, 3, \ldots).$$

In this expression $\bar{\sigma}_{ij}$ should be replaced by Expressions (1.36), and $\bar{u}_i$ by the series (1.58) satisfying Equations (1.59) and the boundary conditions (1.60). Taking into account that, by virtue of Equations (1.59), the boundary conditions (1.60) and the Condition (1.61)

$$-\int_V \frac{\partial}{\partial x_j}\left(\lambda_{ij\alpha\beta}\frac{\partial \varphi_{\alpha n}}{\partial x_\beta}\right) \varphi_{im}\, dV = \Omega_m^2 \int_V \varrho\, \varphi_{im}\, \varphi_{in}\, dV = \Omega_m^2\, \delta_{mn},$$

$$\iint_S \lambda_{ij\alpha\beta}\frac{\partial \varphi_{\alpha n}}{\partial x_\beta} n_j\, \varphi_{im}\, dS = 0,$$

we obtain the system of equations

$$\frac{d^2 f_m}{dt^2} + \Omega_m^2 \sum_n (\delta_{mn} + \beta\, b_{mn}) f_n = 0 \qquad (1.65)$$

$$(m = 1, 2, 3, \ldots).$$

Here

$$b_{mn} = \frac{1}{\Omega_m^2}\left[\int\limits_V \sigma_{jk}\frac{\partial \varphi_{im}}{\partial x_j}\frac{\partial \varphi_{in}}{\partial x_k}\,dV - \int\limits_V \bar{X}_i(\varphi_{in})\,dV - \right.$$

$$\left. - \iint\limits_S \bar{p}_i(\varphi_{in})\,\varphi_{im}\,dS\right]. \tag{1.66}$$

Thus we have arrived at a system of ordinary linear differential equations in the generalized coordinates $f_n(t)$, each of which characterizes the contribution of the n-th mode of oscillation to the motion of the system about its position of undisturbed equilibrium.

We shall consider briefly the structure of the coefficients (1.66). In the case of a "dead" load $\bar{X}_i = \bar{p}_i = 0$; if we set $b_{mn} = a_{mn}$ for this case, we obtain:

$$a_{mn} = \frac{1}{\Omega^2}\int\limits_V \sigma_{jk}\frac{\partial \varphi_{im}}{\partial x_j}\frac{\partial \varphi_{in}}{\partial x_k}\,dV. \tag{1.67}$$

It will be noted that the matrix with elements $a_{mn}\Omega_m^2$ is in this case symmetrical.

In the case of "follower" forces

$$\bar{X}_i = X_k\frac{\partial \bar{u}_i}{\partial x_k}, \qquad \bar{p}_i = p_k\frac{\partial \bar{u}_i}{\partial x_k}$$

and consequently

$$b_{mn} = a_{mn} - \frac{1}{\Omega_m^2}\left[\int\limits_V \frac{\partial \sigma_{jk}}{\partial x_j}\frac{\partial \varphi_{in}}{\partial x_k}\varphi_{im}\,dV + \iint\limits_S \sigma_{jk}\frac{\partial \varphi_{in}}{\partial x_k}\varphi_{im}n_j\,dS\right].$$

Taking into account the equilibrium equations and boundary conditions which are satisfied by the undisturbed components σ_{ij}, X_i and p_i:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = 0, \qquad \sigma_{ij}n_j = p_i,$$

we find after rearrangement that

$$b_{mn} = -\frac{1}{\Omega_m^2}\int\limits_V \sigma_{jk}\varphi_{im}\frac{\partial^2 \varphi_{in}}{\partial x_k^2}\,dV. \tag{1.68}$$

The matrix $\Omega_m^2 b_{mn}$ is not, in general, symmetrical, which is an indication of the non-self-adjoint nature of the problem.

1.13. EVALUATION OF COEFFICIENTS FOR CERTAIN PARTICULAR SYSTEMS

Making use of Formulae (1.67) and (1.68) and introducing further geometrical hypotheses, we can obtain expressions for the coefficients a_{mn} and b_{mn} for various cases of rods, plates and shells. As a very simple example, we shall consider a thin elastic rod with an inextensible axis, under the action of an axial load and able to lose stability by bending in the plane $z\,O\,y$ (Fig. 23). Within the framework of the elementary theory of bending, the stress tensor and the displacement vector for the undisturbed state for such a rod can be written as

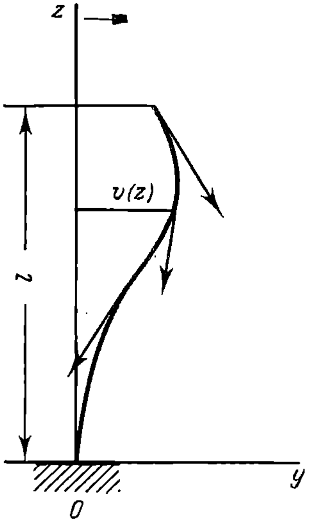


Fig. 23

$$\sigma_{ij} = \begin{Vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \dfrac{N(z)}{F} \end{Vmatrix}, \qquad \bar{u}_j = \begin{Vmatrix} 0 \\ v(z) \\ -y\dfrac{dv(z)}{dz} \end{Vmatrix}.$$

Here $N(z)$ is the longitudinal force, positive for tension, F is the cross-sectional area and $v(z)$ is the deflection of the axis of the rod. Substituting these expressions into Formula (1.67), we find that:

$$a_{mn} = \frac{1}{\Omega_m^2}\int_0^l N(z)\frac{dv_m}{dz}\,\frac{dv_n}{dz}\,dz, \tag{1.69}$$

where l is the length of the rod, $v_m(z)$ are modes of natural oscillations of the rod. It will be noted that Formula (1.61) now leads to the condition

$$\int_0^l \varrho F v_m v_n \, dz + \int_0^l \varrho J_x \frac{dv_m}{dz} \frac{dv_n}{dz} dz = \delta_{mn}, \tag{1.70}$$

where J_x is the second moment of area of the section about the axis $O x$. The second term evidently takes into account rotatory energy. For thin rods its effect is extremely small and in future we shall ignore it. We shall also ignore the analogous terms for plates and shells.

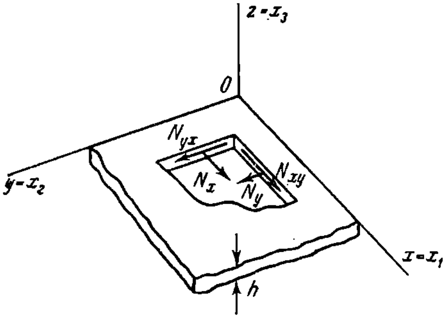


Fig. 24

Formula (1.68) gives[1]

$$b_{mn} = -\frac{1}{\Omega_m^2} \int_0^l N(z) \, v_m \frac{d^2 v_n}{dz^2} dz. \tag{1.71}$$

In the case of a plate of constant thickness h under load in its middle plane $x O y$ (Fig. 24), we have

$$\sigma_{jk} = \left\| \begin{matrix} \frac{N_x}{h} & \frac{N_{xy}}{h} & 0 \\ \frac{N_{yx}}{h} & \frac{N_y}{h} & 0 \\ 0 & 0 & 0 \end{matrix} \right\|, \qquad \bar{u}_j = \left\| \begin{matrix} -z \frac{\partial w(x, y)}{\partial x} \\ -z \frac{\partial w(x, y)}{\partial y} \\ w(x, y) \end{matrix} \right\|,$$

---

[1] Formulae (1.69) and (1.71) were obtained by a different method in the book: BOLOTIN, V. V. *Dynamic Stability of Elastic Systems* (Dinamicheskaia ustoichivost' uprugikh sistem) Gostekhizdat, 1956.

where $w(x, y)$ is the deflection of the middle surface. Applying Formulae (1.67), we obtain

$$a_{mn} = \frac{1}{\Omega_m^2} \iint_S \left[N_x \frac{\partial w_m}{\partial x} \frac{\partial w_n}{\partial x} + \right.$$

$$\left. + N_{xy} \left(\frac{\partial w_m}{\partial x} \frac{\partial w_n}{\partial y} + \frac{\partial w_m}{\partial y} \frac{\partial w_n}{\partial x} \right) + N_y \frac{\partial w_m}{\partial y} \frac{\partial w_n}{\partial y} \right] dS,$$

where dS is an element of area of the middle plane S. Similarly, in the case of "follower" forces we obtain:

$$b_{mn} = -\frac{1}{\Omega_m^2} \iint_S w_m \left(N_x \frac{\partial^2 w_n}{\partial x^2} + 2N_{xy} \frac{\partial^2 w_n}{\partial x \, \partial y} + N_y \frac{\partial^2 w_n}{\partial y^2} \right) dS.$$

The condition of orthogonality and normalization (1.61) can now be written in the form:

$$\iint_S \varrho \, h \, w_m \, w_n \, dS + \iint_S \frac{\varrho \, h^3}{12} \left(\frac{\partial w_m}{\partial x} \frac{\partial w_n}{\partial x} + \frac{\partial w_m}{\partial y} \frac{\partial w_n}{\partial y} \right) dS = \delta_{mn}.$$

The second integral takes into account the rotational inertia, and for thin plates can be discarded.

If curvilinear coordinates are used the corresponding formulae are obtained by replacing the ordinary differentiation by tensorial differentiation:

$$a_{mn} = \frac{1}{\Omega_m^2} \iint_S N^{\alpha\beta} \, \nabla_\alpha \, w_m \, \nabla_\beta \, w_n \, dS,$$

$$b_{mn} = -\frac{1}{\Omega_m^2} \iint_S N^{\alpha\beta} \, w_m \, \nabla_\alpha \, \nabla_\beta \, w_n \, dS.$$

We shall now go very briefly into the question of evaluating the matrix elements a_{mn} and b_{mn} for thin shells. Let x^α ($\alpha = 1, 2$) be the curvilinear coordinates of the middle surface, x^3 a coordinate measured along the normal to the middle surface (Fig. 25). The membrane state of a shell can be described by the tensor of tangential stresses $N_{\alpha\beta}$, and the displacements of the middle surface as the shell deviates from the membrane state of equilibrium, by the displacement vector v_i ($i = 1, 2, 3$). Making use of the hypotheses of the theory of thin shells, we obtain the following expressions for the displacement vector $\bar{u}_i$ for points in the shell:

$$\bar{u}_i = \left\| \begin{matrix} v_1 - z\,\nabla_1 v_3 \\ v_2 - z\,\nabla_2 v_3 \\ v_3 \end{matrix} \right\|.$$

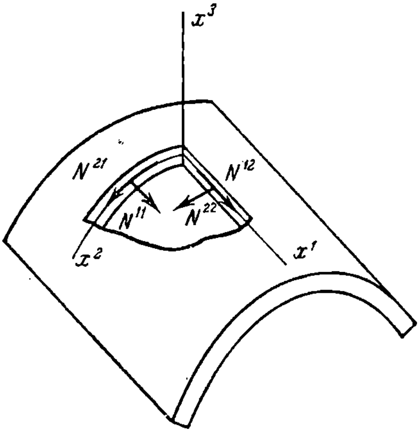


Fig. 25

Thus

$$a_{mn} = \frac{1}{\Omega_m^2} \iint\limits_S N^{\alpha\beta}\, \nabla_\alpha v_{im}\, \nabla_\beta v_{in}\, dS,$$

$$b_{mn} = -\frac{1}{\Omega_m^2} \iint\limits_S N^{\alpha\beta}\, v_{im}\, \nabla_\alpha \nabla_\beta v_{in}\, dS \qquad \left(\begin{matrix} \alpha, \beta = 1, 2 \\ i = 1, 2, 3 \end{matrix}\right).$$

Here v_{im} are the components of the displacement vector of the middle surface for the m-th mode of natural oscillations, dS is an element of area of the middle surface S. If rotational inertia forces are ignored, Condition (1.61) becomes

$$\iint\limits_S \varrho\, h\, v_{im}\, v_{in}\, dS = \delta_{mn}.$$

As a final example, let us consider the problem of the stability of the plane form of bending of a bar under the action of a transverse load in its plane of maximum stiffness. In order not to complicate the computations, we shall confine our attention to the case of a solid bar with two axes of symmetry (Fig. 26). We shall assume that at every section the external surface loading has a resultant lying in one of the planes of symmetry (a plane of maximum stiffness) and acting in a direction perpendicular to the axis of the bar. Then the initial state of stress, the stability of which is to be investigated, is given by

$$\sigma_{jk} = \left\| \begin{matrix} 0 & 0 & 0 \\ 0 & \sigma_{yy} & \tau_{yz} \\ 0 & \tau_z & \sigma_{zz} \end{matrix} \right\|,$$

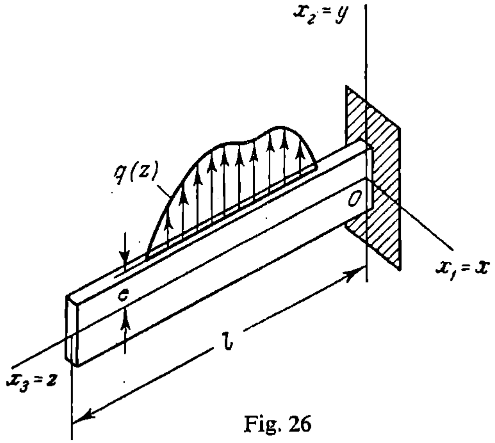


Fig. 26

where, from the elementary theory of bending

$$\sigma_{zz} = \frac{L_x}{J_x} y, \qquad \tau_{yz} = \tau_{zy} = \frac{Q_y S_x}{J_x b}$$

(here, with the exception of the bending moment on cross-sections of the bar, which is denoted by L_x, the usual notations have been adopted and will be used from now on). The stress distribution σ_{yy} depends on the way in which the external load is applied, but it will be shown later that in this case it is not essential to know this distribution.

We shall first determine the matrix elements a_{mn} for the case when the external load acts in a fixed direction. From Formula (1.67)

$$a_{mn} = \frac{1}{\Omega_m^2} \int_0^l dz \int_F \left(\sigma_{yy} \frac{\partial \varphi_{im}}{\partial y} \frac{\partial \varphi_{in}}{\partial y} + \sigma_{zz} \frac{\partial \varphi_{im}}{\partial z} \frac{\partial \varphi_{in}}{\partial z} + \right.$$

$$\left. + \tau_{yz} \frac{\partial \varphi_{im}}{\partial y} \frac{\partial \varphi_{in}}{\partial z} + \tau_{zy} \frac{\partial \varphi_{im}}{\partial z} \frac{\partial \varphi_{in}}{\partial y} \right) dF. \qquad (1.72)$$

Here the summation is carried out for $i = 1, 2, 3$.

The displacement vector for a point on the bar is

$$u_i \doteq \left\| \begin{matrix} u - y\theta \\ v + x\theta \\ w - x\dfrac{du}{dz} - y\dfrac{dv}{dz} \end{matrix} \right\|, \tag{1.73}$$

where $u(z)$, $v(z)$, $w(z)$ are the displacements of points lying on the axis of the bar along the coordinate axes $O\,x$, $O\,y$, $O\,z$; $\theta(z)$ is the angle of twist, the positive direction of which is shown in Fig. 27. The effect of warping is ignored.

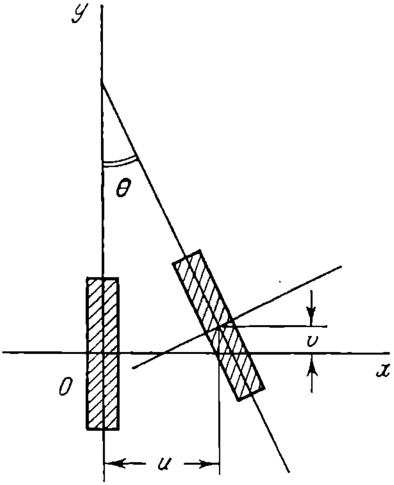


Fig. 27

Loss of stability of the plane form of bending can be described by the displacements $u(z)$ and $\theta(z)$, and we shall therefore derive an expression for the vectors $\varphi_{i\,m}$. Let $u_m(z)$ and $\theta_m(z)$ be the m-th modes of natural oscillations of the unloaded bar, and let $\Omega_{x\,m}$ and $\Omega_{\theta\,m}$ be the corresponding natural frequencies (if the section has two axes of symmetry the natural bending and torsional oscillations take place independently of each other). For convenience we shall set

$$u_m(z) = \varphi_{2m-1}(z), \qquad \Omega_{x\,m} = \Omega_{2m-1},$$
$$\theta_m(z) = \varphi_{2m}(z), \qquad \Omega_{\theta\,m} = \Omega_{2m}$$
$$(m = 1, 2, 3, \ldots).$$

Then in accordance with (1.73) the vectors $\varphi_{i\,m}$ can be written in the form

$$\varphi_{i,\,2m-1} = \left\| \begin{matrix} \varphi_{2m-1} \\ 0 \\ -y\,\dfrac{d\varphi_{2m-1}}{dz} \end{matrix} \right\|; \qquad \varphi_{i.\,2m} = \left\| \begin{matrix} -y\,\varphi_{2m} \\ x\,\varphi_{2m} \\ 0 \end{matrix} \right\|. \tag{1.74}$$

It follows from Formula (1.61) that the functions φ_{2m-1} and φ_{2m} must satisfy the conditions of normalization

$$\int\limits_0^l \varrho\, F \varphi_{2m-1}^2\, dz = 1, \qquad \int\limits_0^l \varrho\, J_z\, \varphi_{2m}^2\, dz = 1, \tag{1.75}$$

where J_z is the polar second moment of area of the section.

Substituting Expressions (1.74) into Formulae (1.72) we obtain

$$a_{mn} = \begin{cases} \dfrac{1}{\Omega_m^2} \displaystyle\int\limits_0^l dz \int\limits_F \left[\sigma_{zz}\left(\frac{d\varphi_m}{dz}\,\frac{d\varphi_n}{dz} + y^2\,\frac{d^2\varphi_m}{dz^2}\,\frac{d^2\varphi_n}{dz^2}\right) - \right. \\ \qquad\qquad \left. - \tau_{yz}\, y \left(\dfrac{d\varphi_m}{dz}\,\dfrac{d^2\varphi_n}{dz^2} + \dfrac{d^2\varphi_m}{dz^2}\,\dfrac{d\varphi_n}{dz}\right)\right] dF, \\ \text{if } m \text{ and } n \text{ are odd numbers;} \\ \dfrac{1}{\Omega_m^2} \displaystyle\int\limits_0^l dz \int\limits_F \left[\sigma_{zz}(x^2+y^2)\,\frac{d\varphi_m}{dz}\,\frac{d\varphi_n}{dz} + \sigma_{yy}\,\varphi_m\,\varphi_n - \right. \\ \qquad\qquad \left. - \tau_{yz}\, y \left(\varphi_m \dfrac{d\varphi_n}{dz} + \varphi_n \dfrac{d\varphi_m}{dz}\right)\right] dF, \\ \text{if } m \text{ and } n \text{ are even numbers;} \\ -\dfrac{1}{\Omega_m^2} \displaystyle\int\limits_0^l dz \int\limits_F \left(\sigma_{zz}\, y\, \frac{d\varphi_m}{dz}\,\frac{d\varphi_n}{dz} + \tau_{yz}\,\frac{d\varphi_m}{dz}\,\varphi_n\right) dF, \\ \text{if } m \text{ is odd and } n \text{ is even;} \\ -\dfrac{1}{\Omega_m^2} \displaystyle\int\limits_0^l dz \int\limits_F \left(\sigma_{zz}\, y\, \frac{d\varphi_m}{dz}\,\frac{d\varphi_n}{dz} + \tau_{yz}\,\varphi_m\,\frac{d\varphi_n}{dz}\right) dF, \\ \text{if } m \text{ is even and } n \text{ is odd.} \end{cases}$$

But according to the elementary theory of bending

$$\int\limits_F \sigma_{zz}\, dF = 0, \qquad \int\limits_F \sigma_{zz}\, y\, dF = L_x, \qquad \int\limits_F \tau_{yz}\, dF = Q_y = \frac{dL_x}{dz},$$

and, moreover, from considerations of symmetry

$$\int\limits_F \tau_{yz}\, y\, dF = 0, \qquad \int\limits_F \sigma_{zz}\, x^2\, dF = \int\limits_F \sigma_{zz}\, y^2\, dF = 0.$$

It now remains to express the integral

$$\int\limits_F \sigma_{yy}\, dF.$$

in terms of the external forces. To do so we use the well-known formula[1] for the mean values of the components of the stress tensor, according to which

$$\int\limits_V \sigma_{ik}\, dV = \iint\limits_S p_i\, x_k\, dS.$$

Considering an element of unit length of the bar, we find

$$\int\limits_F \sigma_{yy}\, dF = \int\limits_\Gamma p_y\, y\, d\Gamma,$$

where p_y is the component of the surface forces along the axis $O\,y$, Γ is the contour bounding the cross-section. The resultant of the surface forces per unit length of the bar and its height above the axis $O\,z$ are given by

$$\int\limits_\Gamma p_y\, d\Gamma = -q, \qquad e = \frac{\int\limits_\Gamma p_y\, y\, d\Gamma}{\int\limits_\Gamma p_y\, d\Gamma}. \tag{1.76}$$

Then

$$\int\limits_F \sigma_{yy}\, dF = -q\, e = \frac{d^2 L_x}{dz^2}\, e.$$

---

[1] LANDAU, L. D. and LIFSHITS, E. M. *Mechanics of Continuous Media* (Mekhanika sploshnykh sred), Gostekhizdat, 1954.

Thus

$$a_{mn} = \begin{cases} 0, & \text{if } m \text{ and } n \text{ are odd numbers;} \\ -\dfrac{1}{\Omega_m^2}\displaystyle\int_0^l q\, e\, \varphi_m \varphi_n\, dz, & \text{if } m \text{ and } n \text{ are even numbers;} \\ -\dfrac{1}{\Omega_m^2}\displaystyle\int_0^l \frac{d\varphi_m}{dz}\frac{d}{dz}(L_x \varphi_n)\, dz, & \text{if } m \text{ is odd and } n \text{ is even;} \\ -\dfrac{1}{\Omega_m^2}\displaystyle\int_0^l \frac{d\varphi_n}{dz}\frac{d}{dz}(L_x \varphi_m)\, dz, & \text{if } m \text{ is even and } n \text{ is odd.} \end{cases} \tag{1.77}$$

Let us now consider the case of a follower load. Formula (1.66) in this case becomes

$$b_{mn} = a_{mn} + \int_0 dz \int_\Gamma p_y \frac{d\varphi_{in}}{\partial z} \varphi_{im}\, d\Gamma.$$

Substituting Expressions (1.74) and (1.77) into this formula, and making use of (1.76), we obtain.[1]

$$b_{mn} = \begin{cases} 0, \\ \text{if } m \text{ and } n \text{ are either both even or both odd} \\ -\dfrac{1}{\Omega_m^2}\left[\displaystyle\int_0^l \frac{d\varphi_m}{dz}\frac{d}{dz}(L_x \varphi_n)\, dz + \int_0^l q\, \varphi_m \varphi_n\, dz\right], \\ \text{if } m \text{ is odd and } n \text{ is even;} \\ -\dfrac{1}{\Omega_m^2}\displaystyle\int_0^l \frac{d\varphi_n}{dz}\frac{d}{dz}(L_x \varphi_m)\, dz, \\ \text{if } m \text{ is even and } n \text{ is odd.} \end{cases} \tag{1.78}$$

---

[1] Formulae (1.77) and (1.78) were first derived from the equations of thin-walled bars in the paper: BOLOTIN, V. V. On the vibrations and stability of bars under the action of non-conservative forces. *Sb.* "*Kolebaniia v turbomashinakh*", Izd-vo. Akad. Nauk USSR, 1959.

Note that the matrix b_{mn} for this problem has a peculiar structure:

$$b_{mn} = \begin{Vmatrix} 0 & b_{12} & 0 & b_{14} & \ldots \\ b_{21} & 0 & b_{23} & 0 & \ldots \\ 0 & b_{32} & 0 & b_{34} & \ldots \\ b_{41} & 0 & b_{43} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{Vmatrix}.$$

1.14. INVESTIGATION OF STABILITY

Let us suppose that a system is subjected to conservative forces specified to the accuracy of a parameter α, and in addition, by forces without potential (for example "follower" forces) specified to the accuracy of a parameter β. The corresponding matrices of the coefficients, as before, will be denoted by a_{jk} and b_{jk}. Furthermore, we shall assume that dissipative forces, proportional to the generalized velocities df_j/dt, act on the body. We shall also assume that there is no dissipative relation between the various modes of oscillation. The dissipative forces will then be represented by a diagonal matrix with elements ε_j. The system of equations which describes small oscillations about the equilibrium position is of the form

$$\frac{d^2 f_j}{dt^2} + \varepsilon_j \frac{df_j}{dt} + \Omega_j^2 \left(f_j + \alpha \sum_{k=1}^{n} a_{jk} f_k + \beta \sum_{k=1}^{n} b_{jk} f_k \right) = 0$$

$$(j = 1, 2, \ldots, n). \tag{1.79}$$

Here n is the number of modes of oscillation sufficient for a satisfactory representation of the motion of an elastic system about its undisturbed position of equilibrium.

Equations (1.79) can, of course, be looked upon as a particular case of the equations of motion of a mechanical system with n degrees of freedom about its equilibrium position:

$$\frac{d^2 f_j}{dt^2} + \sum_{k=1}^{n} A_{jk} \frac{df_k}{dt} + \sum_{k=1}^{n} B_{jk} f_k = 0 \qquad (j = 1, 2, \ldots, n).$$

Representing the matrix B_{jk} as the sum of a symmetric and an antisymmetric matrix

$$B_{jk} = \frac{B_{jk} + B_{kj}}{2} + \frac{B_{jk} - B_{kj}}{2}$$

corresponds to splitting the forces which depend on position into *potential* forces and *essentially non-potential forces*[1]. This, however, differs in general from splitting the matrix in the following way:

$$B_{jk} = \alpha\,\Omega_j^2\,a_{jk} + \beta\,\Omega_j^2\,b_{jk}.$$

In the case of "follower" forces, therefore, the matrix $\Omega_j^2\,b_{jk}$ is not necessarily antisymmetric. Since the basic problem of the theory of elastic stability is to investigate the conditions of stability (or instability) with respect to load parameters, it is more convenient to write the equations in the form (1.79).

In order to find the parameters which correspond to bifurcation of the forms of equilibrium, we have the equation

$$|\delta_{jk} + \alpha\,a_{jk} + \beta\,b_{jk}| = 0, \tag{1.80}$$

which is obtained by equating to zero the determinant of the coefficients of the homogeneous algebraic system

$$f_j + \alpha \sum_{k=1}^{n} a_{jk} f_k + \beta \sum_{k=1}^{n} b_{jk} f_k = 0 \qquad (j = 1, 2, \ldots, n).$$

Since the matrix a_{jk} can be made symmetrical by pre-multiplying by the matrix $\Omega_j\,\delta_{jk}$, all its eigenvalues given by the equation

$$|\delta_{jk} + \alpha\,a_{jk}| = 0, \tag{1.81}$$

are real. This, in general, does not hold for the matrix b_{jk}. Suppose that the parameter α has a fixed value less than the minimum eigenvalue α^*. Then for $\beta = 0$ the equilibrium of the system is stable. The problem of the behavior of the system as β increases from zero to a particular value cannot be solved in a general form, since in this case everything depends on the structure of the matrices a_{jk} and b_{jk}.

We shall now consider small oscillations of the system about its equilibrium position, assuming initially that there is no damping. Equations (1.79) are satisfied if we set

$$f_j = F_j\,e^{i\Omega t},$$

---

[1] These forces are often described in other terms. For instance: as "circulatory forces" (Ziegler, H. *On the stability of elastic systems.* see ref. on p. 81), as "pseudo-gyroscopic forces" (Obmorshev, N. A. *Oscillations and stability of motion in machines*, MVTU, 1952) and as "forces of radial correction" (Merkin, D. R. *Gyroscopic Systems* (Giroskopicheskie sistemy), Gostekhizdat, 1956).

where F_j are constants. The equation

$$\left|\left(1-\frac{\Omega^2}{\Omega_j^2}\right)\delta_{jk}+\alpha\, a_{jk}+\beta\, b_{jk}\right|=0 \tag{1.82}$$

can be used to give the fequencies of oscillation Ω. If $\alpha = \beta = 0$, the frequencies are real and equal to Ω_j. They remain real for $\alpha \neq 0$, $\beta \neq 0$, provided these parameters are sufficiently small and there are no multiple roots amongst the frequencies Ω_j. This means that for sufficiently small values of the load parameters all the characteristic exponents $s = i\Omega$ lie on the imaginary axis. If $\beta = 0$ all Ω^2 are real, and consequently the characteristic exponents can enter the right-hand half-plane only after passing through the value of $s = i\Omega = 0$. This corresponds to the static form of loss of stability. If $\beta \neq 0$, then both the static and the dynamic (oscillatory) forms of instability are possible. The form of instability depends both on the structure of the matrices a_{jk}, b_{jk} and $\Omega_j^2\, \delta_{jk}$, and on the relation between α and β. Each case must be investigated by itself. In practice this amounts to finding the real roots Ω of Equation (1.82) as functions of the parameters α and β, and finding relations between these parameters for which the real roots become multiple and then complex with any further change in the parameters.

When damping exists we shall try to find a solution to Equation (1.79) in the form

$$f_j = F_j\, e^{st} \qquad (j = 1, 2, \ldots, n).$$

If all $\varepsilon_j > 0$, which corresponds to a system with complete dissipation, then for $\alpha = \beta = 0$ all the roots of the characteristic equation

$$|(s^2+\varepsilon_j s+\Omega_j^2)\,\delta_{jk}+\Omega_j^2(\alpha\, a_{jk}+\beta\, b_{jk})| = 0 \tag{1.83}$$

lie in the left-hand half-plane. The problem is to find a region of stability in the plane of the parameters α and β, or in some other space of the parameters. This problem for small values of n can be solved by means of algebraic stability criteria, for example, Hurwitz's criterion. In more complicated cases other methods can be used, which are mainly the results of detailed investigations in the field of automatic control.

The following question is of considerable interest. Let us suppose that a region of stability has been found based on two assumptions, the first ignoring damping and the second taking it into account. In the first case stability means that all the characteristic exponents were found to lie on the imaginary axis, and in the second case they were all in the left-hand half-plane of the complex variable. The question immediately arises whether the region of stability for the first case includes that for the second case. In other words, does the addition of dissipative forces stabilize the undisturbed equilibrium or not? For a system in equilibrium under the action of potential forces the answer to this question is affirmative. The answer is to be found, in fact, in the well-known theorem of Kelvin, which states that the addition of dissipative forces with complete dissipation ensures asymptotic stability of the undisturbed equilibrium. In the case of non-conservative systems this, in general, is not so: here the addition of dissipative forces can in certain cases have a destabilizing effect.

In this connection another very interesting question arises. Suppose that the limit of stability has been found for the case when damping occurs, and that all the coefficients ε_j are then made to tend to zero. Will the limit of stability corresponding to a gradually vanishing damping effect coincide in the limit with that found on the assumption that there is no damping? In the case of conservative forces the answer is that it will. However, if the forces have no potential, the limit of stability found for a system with a very small damping effect which gradually vanishes, and that found for a system without damping, do not, in general, coincide.

1.15. Example. System with two Degrees of Freedom

All the principal features of non-conservative systems can be illustrated by a very simple example of a system with two degrees of freedom with the matrices

$$a_{jk} = \left\| \begin{matrix} 0 & a_{12} \\ a_{21} & 0 \end{matrix} \right\|, \qquad b_{jk} = \left\| \begin{matrix} 0 & b_{12} \\ b_{21} & 0 \end{matrix} \right\|.$$

To a first approximation the problem of the stability of the plane form of bending of a bar under the action of follower forces reduces to a system of this sort. An analogous system is obtained in an approximate investigation of problems on the stability of the rectilinear form of a bar in torsion, on the stability of plates in supersonic gas flow and on the stability of the rectilinear form of a shaft rotating at a constant speed under the action of internal friction.

It should first be noted that the Euler critical values of the parameters α and β are

$$\alpha_* = \pm \frac{1}{\sqrt{(a_{12} a_{21})}}, \qquad \beta_* = \pm \frac{1}{\sqrt{(b_{12} b_{21})}}. \tag{1.84}$$

Since we are principally concerned here with the case when the static form of loss of stability is impossible under the action of loads with the parameter β, we can assume that $b_{12} b_{21} < 0$. Conversely, $a_{12} a_{21} > 0$.

We shall consider the case when there is no damping. Suppose initially that $\alpha = 0$. Equation (1.82) becomes

$$\begin{vmatrix} \Omega_1^2 - \Omega^2 & \beta \Omega_1^2 b_{12} \\ \beta \Omega_2^2 b_{21} & \Omega_2^2 - \Omega^2 \end{vmatrix} = 0.$$

Using the notations

$$\frac{\Omega_j}{\Omega_0} = \omega_j, \qquad \frac{\Omega}{\Omega_0} = \omega, \qquad b_{12} b_{21} = -\frac{1}{\beta_0^2}, \tag{1.85}$$

where Ω_0 is the natural frequency (for example, Ω_1), we obtain the equation

$$\omega^4 - \omega^2(\omega_1^2 + \omega_2^2) - \omega_1^2 \omega_2^2 \left(1 + \frac{\beta^2}{\beta_0^2}\right) = 0.$$

Its roots

$$\omega = \pm \frac{1}{\sqrt{2}} \sqrt{\left\{\omega_1^2 + \omega_2^2 \pm \sqrt{\left[(\omega_2^2 - \omega_1^2)^2 - 4\omega_1^2 \omega_2^2 \frac{\beta^2}{\beta_0^2}\right]}\right\}}$$

are real if the expression under the inner square root is positive. Thus the critical value of the parameter β is

$$\beta_* = \beta_0 \frac{\omega_2^2 - \omega_1^2}{2\omega_1 \omega_2}. \tag{1.86}$$

The relation between the non-dimensional frequencies ω and the ratio β/β_0 is shown in Fig. 28. As β increases the frequencies ω approach each other, and at $\beta = \beta_*$ they become multiple and equal to

$$\omega_* = \sqrt{\frac{\omega_1^2 + \omega_2^2}{2}}.$$

With further increase in β two of the four characteristic exponents $\sigma = i\omega$ pass into the right-hand half-plane (see Fig. 29, which shows only the upper half-plane). Note that if the partial frequencies of the unloaded system are equal ($\omega_1 = \omega_2$), then, as will be seen from Formula (1.86), the equilibrium of the system is unstable no matter how small the values of β. This paradoxical result is due to the fact that damping has not been taken into account.

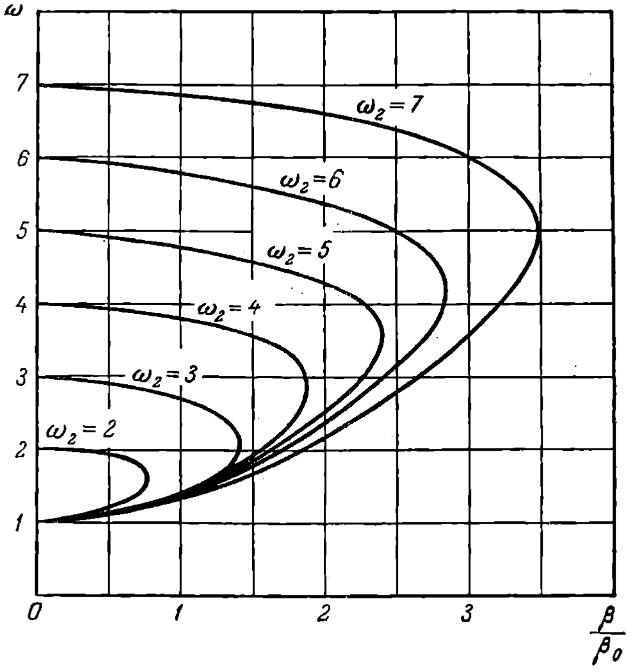


Fig. 28

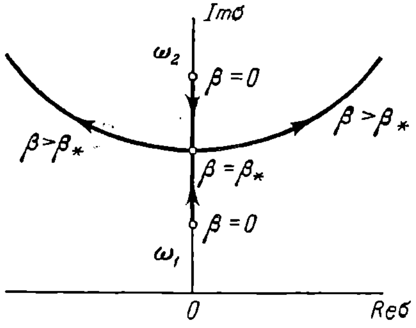


Fig. 29

Let us assume now that both parameters α and β are non-zero. Equation (1.82) becomes

$$\begin{vmatrix} \Omega_1^2 - \Omega^2 & \alpha\, a_{12} + \beta\, b_{12} \\ \alpha\, a_{21} + \beta\, b_{21} & \Omega_2^2 - \Omega^2 \end{vmatrix} = 0.$$

Putting

$$b_{12}\, a_{21} + a_{12}\, b_{21} = \frac{2A}{\alpha_*\, \beta_0},$$

we arrive at the equation

$$\omega^4 - \omega^2(\omega_1^2 + \omega_2^2) + \omega_1^2\, \omega_2^2 \left(1 - \frac{\alpha^2}{\alpha_*^2} + \frac{\beta^2}{\beta_0^2} - \frac{2A\, \alpha\, \beta}{\alpha_*\, \beta_0}\right) = 0.$$

In other words

$$\omega^4 - \omega^2(\omega_1^2 + \omega_2^2) + \omega_1^2\, \omega_2^2 \left[1 + \frac{(\mu + \nu)^2}{(1 + \psi)^2} (\psi^2 - 2A\, \psi - 1)\right] = 0. \tag{1.87}$$

Here

$$\mu = \frac{\alpha}{\alpha_*}, \qquad \nu = \frac{\beta}{\beta_0}, \qquad \psi = \frac{\nu}{\mu}, \tag{1.88}$$

i.e. the parameter μ characterizes the conservative component of the loading, ν characterizes the non-conservative component and ψ their ratio.

The roots of Equation (1.87) are given by the formula

$$\omega^2 = \frac{1}{2}\Bigg[\omega_1^2 + \omega_2^2 \pm \\ \pm \sqrt{\left((\omega_2^2 - \omega_1^2)^2 - 4\omega_1^2\, \omega_2^2 \frac{(\mu + \nu)^2}{(1 + \psi)^2} (\psi^2 - 2A\, \psi - 1)\right)}\Bigg]. \tag{1.89}$$

Two of the roots of Equation (1.87) vanish when

$$1 + \frac{(\mu + \nu)^2}{(1 + \psi)^2} (\psi^2 - 2A\, \psi - 1) = 0.$$

Thus the Euler critical value of the paramter $\mu + \nu$ is

$$(\mu + \nu)_* = \frac{1 + \psi}{\sqrt{(1 + 2A\, \psi - \psi^2)}}. \tag{1.90}$$

An oscillatory type of instability will occur if the expression under the square root sign in Formula (1.89) becomes negative. Thus

$$(\mu + \nu)_* = \frac{\omega_2^2 - \omega_1^2}{2\omega_1\, \omega_2} \frac{1 + \psi}{\sqrt{(\psi^2 - 2A\, \psi - 1)}}. \tag{1.91}$$

Transition from one type of stability to the other takes place when

$$\psi = \psi_0 = A + \sqrt{(A^2 + 1)}. \qquad (1.92)$$

A graph representing Relations (1.90) and (1.91) is shown in Fig. 30. It has been assumed that $A = 1.07$ and $\omega_2 = 2.00\,\omega_1$. If ψ is not too large, addition of non-conservative forces increases the stability of undisturbed equilibrium[1].

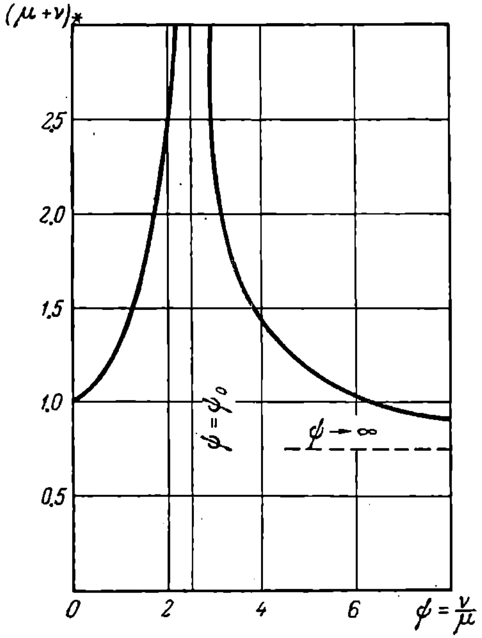


Fig. 30

1.16. Effect of Dissipative Forces on Stability

In order to illustrate the peculiar effect of dissipative forces in the presence of non-conservative forces which depend on position, let us consider the following system of equations

$$\frac{d^2 f_j}{dt^2} + \varepsilon_j \frac{df_j}{dt} + \Omega_j^2 \Big(f_j + \beta \sum_{k=1,2} b_{jk} f_k\Big) = 0 \qquad (j = 1, 2).$$

---

[1] In gyroscopic instruments non-conservative forces produced, for example, by the pressure of a jet are used to increase stability ("radial correction forces", see Merkin, D. R.; reference on page 73).

Setting

$$\frac{s}{\Omega_0} = \sigma, \qquad \frac{\Omega_j}{\Omega_0} = \omega_j, \qquad \frac{\varepsilon_j}{\Omega_0} = g_j, \tag{1.93}$$

where Ω_0 is some frequency parameter, we can write the characteristic equation in the form

$$\begin{vmatrix} \sigma^2 + g_1 \sigma + \omega_1^2 & \beta\, \omega_1^2 b_{12} \\ \beta\, \omega_2^2 b_{21} & \sigma^2 + g_2 \sigma + \omega_2^2 \end{vmatrix} = 0.$$

Expanding the determinant and making use of the notation (1.85) for β_0^2, we obtain

$$\sigma^4 + \sigma^3(g_1 + g_2) + \sigma^2(\omega_1^2 + \omega_2^2 + g_1 g_2) + \\ + \sigma(g_2 \omega_1^2 + g_1 \omega_2^2) + \omega_1^2 \omega_2^2 \left(1 + \frac{\beta^2}{\beta_0^2}\right) = 0. \tag{1.94}$$

We have obtained a fourth-order algebraic equation

$$a_0 \sigma^4 + a_1 \sigma^3 + a_2 \sigma^2 + a_3 \sigma + a_4 = 0,$$

all the roots of which lie in the left-hand half-plane if all its coefficients

$$a_0 > 0, \quad a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_4 > 0$$

and if in addition the condition

$$(a_1 a_2 - a_0 a_3)\, a_3 - a_1^2 a_4 > 0. \tag{1.95}$$

is satisfied. If g_1 and g_2 are positive, all the coefficients of Equation (1.94) are positive. There remains the final condition, which assumes the form

$$\beta^2 < \beta_0^2 \times \\ \times \left\{\frac{[(g_1 + g_2)(\omega_1^2 + \omega_2^2 + g_1 g_2) - (g_2 \omega_1^2 + g_1 \omega_2^2)](g_2 \omega_1^2 + g_2 \omega_1^2)}{(g_1 + g_2)^2 \omega_1^2 \omega_2^2} - 1\right\}.$$

Thus the critical value of the parameter β is:

$$\beta_{**} = \frac{\beta_0 \sqrt{(g_1 g_2)}}{(g_1 + g_2)\, \omega_1 \omega_2} \sqrt{[(\omega_2^2 - \omega_1^2)^2 + g_2^2 \omega_1^2 + g_1 g_2(\omega_1^2 + \omega_2^2) + g_1^2 \omega_2^2]}. \tag{1.96}$$

If in this formula we set

$$\eta = \frac{g_2}{g_1} = \frac{\varepsilon_2}{\varepsilon_1}, \qquad g_1 = g \tag{1.97}$$

and take into account Formula (1.86) for the critical parameter β_*, found without taking damping into account, we obtain

$$\beta_{**} = \beta_* \frac{2\sqrt{\eta}}{1+\eta}\left\{1 + O\left[\frac{g^2}{(\omega_2^2 - \omega_1^2)^2}\right]\right\}.$$

If slight damping takes place and if the partial frequencies are not too close together, we can set

$$\beta_{**} \approx \beta_* \frac{2\sqrt{\eta}}{1+\eta}. \tag{1.98}$$

If we consider the right-hand side as a function of the parameter η, we arrive at the conclusion that it reaches a maximum value of β_* when $\eta = 1$. As can be seen from (1.97), this corresponds to the equality of the damping coefficients $\varepsilon_1 = \varepsilon_2$. If $\varepsilon_1 \neq \varepsilon_2$ the critical value β_{**} found for the case when the damping is small and then vanishes is less than the critical value β_* for the case of no damping (Fig. 31). For sufficiently large values of g damping can once more have a stabilizing effect (Fig. 32). The destabilizing effect of friction was first discovered by Ziegler[1] in the simple example of a double pendulum under the action of a follower force. The result given here, which

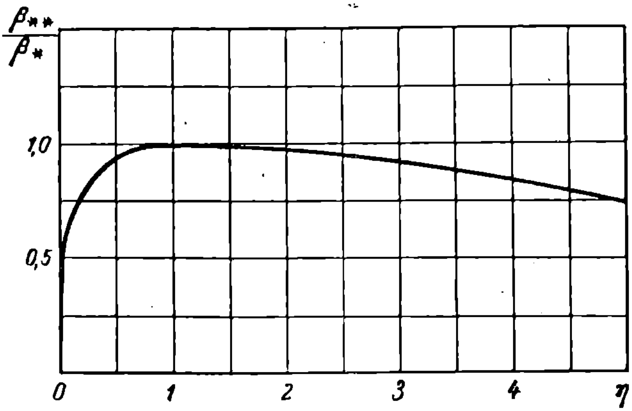


Fig. 31

---

[1] Ziegler, H. Die Stabilitätskriterien der Elastomechanik, *Ing.-Arch.* **20**, No. 1 (1952). Let us note once again that for a system without friction for $\beta < \beta_*$ we have stability in the sense of Kelvin; for a system with friction for $\beta < \beta_{**}$ we have asymptotic stability.

refers to the effect of the relation between the damping coefficients, was derived by the author[1].

The phenomenon of the destabilizing effect is illustrated in Fig. 33, which shows the behavior of the characteristic exponents in the

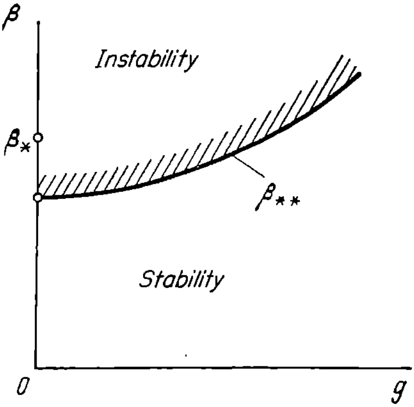


Fig. 32

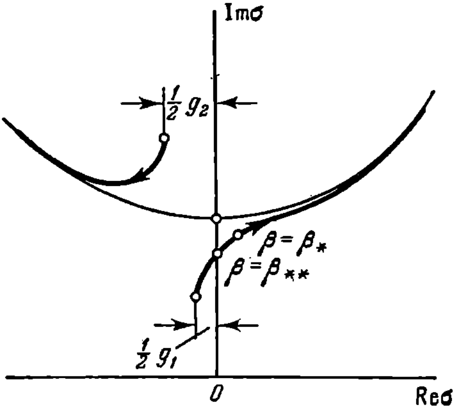


Fig. 33

absence and presence of damping when ε_1 and ε_2 are not equal. If the damping is not severe then for $\varepsilon_1 \neq \varepsilon_2$ one of the exponents leaves the left-hand half-plane at a value β_{**} less than β_*.

---

[1] Bolotin, V. V. *Some problems of the theory of elastic stability*, Trans. 3rd All-Soviet Mathematics Conference, Vol. 1., Izd-vo. Akad. Nauk USSR, 1956.

Let us consider the case when the partial frequencies are close together. We can express Formula (1.96) in the form

$$\beta_{**} = \beta_0 \frac{\sqrt{\eta}}{(1+\eta)\,\omega_1\,\omega_2} \sqrt{\{(\omega_2^2 - \omega_1^2)^2 + g^2[\eta^2\,\omega_1^2 + \eta(\omega_1^2 + \omega_2^2) + \omega_2^2]\}}. \tag{1.99}$$

Here even slight damping might have a stabilizing effect. For instance, in the case of multiple frequencies

$$\beta_{**} = \beta_0\, g \sqrt{\eta} = \beta_0 \sqrt{(g_1\, g_2)}, \tag{1.100}$$

whereas if damping had been ignored we would have obtained the paradoxical result that $\beta_* = 0$.

It can be shown that the same results apply in the more general case of a system with two degrees of freedom, provided certain broad assumptions are made concerning the matrix b_{jk}. Consider the case when $b_{11} > 0$, $b_{22} > 0$ and $b_{12}\, b_{21} > 0$. Then the equation

$$|\delta_{jk} + \beta\, b_{jk}| = 0,$$

obviously has no roots. The critical values for this case, when there is slight damping which then vanishes, can be compared with the corresponding values found on the assumption that there is no damping from the beginning on.

The characteristic equation

$$\begin{vmatrix} \sigma^2 + \sigma\, g_1 + \omega_1^2 + \beta\, \omega_1^2\, b_{11} & \beta\, \omega_1^2\, b_{12} \\ \beta\, \omega_2^2\, b_{21} & \sigma^2 + \sigma\, g_2 + \omega_2^2 + \beta\, \omega_2^2\, b_{22} \end{vmatrix} = 0$$

reduces to the form

$$a_0\, \sigma^4 + a_1\, \sigma^3 + a_2\, \sigma^2 + a_3\, \sigma + a_4 = 0,$$

where

$$\begin{aligned} a_0 &= 1, \\ a_1 &= g(1+\eta), \\ a_2 &= \omega_1^2 + \omega_2^2 + \eta\, g^2 + \beta(\omega_1^2\, b_{11} + \omega^2\, b_{22}), \\ a_3 &= g[\eta\, \omega_1^2 + \omega_2^2 + \beta(\eta\, \omega_1^2\, b_{11} + \omega_2^2\, b_{22})], \\ a_4 &= \omega_1^2\, \omega_2^2[1 + \beta(b_{11} + b_{22}) + \beta^2(b_{11}\, b_{22} - b_{12}\, b_{21})]. \end{aligned}$$

By virtue of the assumptions concerning the properties of the matrix b_{jk}, all the coefficients a_k are positive. Consequently there remains only one non-trivial Hurwitz Condition (1.95), which we can write in the form

$$\Phi(\beta) = A\beta^2 + B\beta + C > 0.$$

Here

$$\begin{aligned} A &= \eta(\omega_1^2 b_{11} - \omega_2^2 b_{22})^2 - (1+\eta)^2 \omega_1^2 \omega_2^2 b_{12} b_{21}, \\ B &= \eta\,\omega_2^2 b_{22}[2(\omega_2^2 - \omega_1^2) + g^2(1+\eta)] - \\ &\quad - \eta\,\omega_1^2 b_{11}[2(\omega_2^2 - \omega_1^2) - \eta\, g^2(1+\eta)], \\ C &= \eta[(\omega_2^2 - \omega_1^2)^2 + g^2(\eta\,\omega_1^2 + \omega_2^2)(1+\eta)]. \end{aligned}$$

We let the damping now tend to zero, keeping η non-zero. This latter condition can be written as

$$\Phi_1(\beta) = A_1\beta^2 + B_1\beta + C_1 > 0, \tag{1.101}$$

where

$$\left.\begin{aligned} A_1 &= (\omega_1^2 b_{11} - \omega_2^2 b_{22})^2 - \left(\frac{1}{\sqrt{\eta}} + \sqrt{\eta}\right)^2 \omega_1^2 \omega_2^2 b_{12} b_{21}, \\ B_1 &= 2(\omega_2^2 - \omega_1^2)(\omega_2^2 b_{22} - \omega_1^2 b_{11}), \\ C_1 &= (\omega_2^2 - \omega_1^2)^2. \end{aligned}\right\} \tag{1.102}$$

Consider now a system in which the damping is identically zero. The characteristic equation is of the form

$$\begin{vmatrix} \sigma^2 + \omega_1^2 + \beta\,\omega_1^2 b_{11} & \beta\,\omega_1^2 b_{12} \\ \beta\,\omega_2^2 b_{21} & \sigma^2 + \omega_2^2 + \beta\,\omega_2^2 b_{22} \end{vmatrix} = 0$$

or

$$\sigma^4 + a_2\sigma^2 + a_4 = 0,$$

where

$$\left.\begin{aligned} a_2 &= \omega_1^2 + \omega_2^2 + \beta(\omega_1^2 b_{11} + \omega_2^2 b_{22}), \\ a_4 &= \omega_1^2 \omega_2^2[1 + \beta(b_{11} + b_{22}) + \beta^2(b_{11} b_{22} - b_{12} b_{21})]. \end{aligned}\right\} \tag{1.103}$$

All the roots of this equation will lie on the imaginary axis if $a_2 > 0, a_4 > 0$ and $a_2^2 - 4a_4 > 0$. The first two conditions are satisfied identically by virtue of the assumptions concerning the properties of the matrix b_{jk}. The last condition can be written in the form

$$\Phi_2(\beta) = A_2\beta^2 + B_2\beta + C_2 > 0, \tag{1.104}$$

where

$$\begin{aligned} A_2 &= (\omega_1^2 b_{11} - \omega_2^2 b_{22})^2 - 4\omega_1^2 \omega_2^2 b_{12} b_{21}, \\ B_2 &= B_1, \quad C_2 = C_1. \end{aligned}$$

We can now write down the difference

$$\Phi_1(\beta) - \Phi_2(\beta) = -\beta^2 \omega_1^2 \omega_2^2 b_{12} b_{21} \left[4 - \left(\sqrt{\eta} + \frac{1}{\sqrt{\eta}}\right)^2\right].$$

It will be seen that the expression in the square brackets, considered as a functions of η, has a maximum value of zero at $\eta = 1$. Thus for all η and β

$$\Phi_1(\beta) - \Phi_2(\beta) \leqq 0, \tag{1.105}$$

provided only that $b_{12} b_{21} < 0$. When $\beta = 0$ we have that, $\Phi_1(0) = C_1 > 0$, $\Phi_2(0) = C_2 > 0$. We now let β vary from zero in a positive direction. Then from the inequality (1.105) it follows that the roots of the equations $\Phi_1(\beta) = 0$ and $\Phi_2(\beta) = 0$, denoted by β_{**} and β_* respectively, satisfy the relation $\beta_{**} \leqq \beta_*$. We obtain an analogous relation for $\beta < 0$: here $|\beta_{**}| \leqq |\beta_*|$ (Fig. 34).

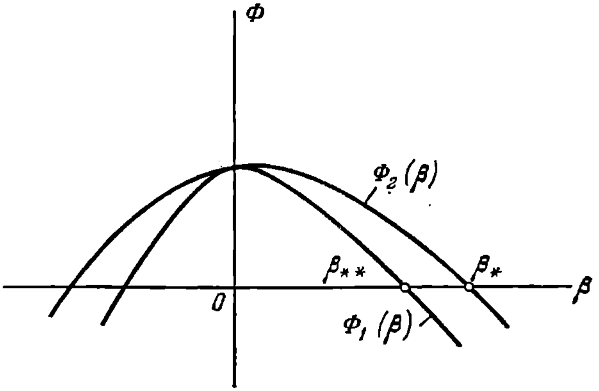


Fig. 34

Thus, if $b_{11} > 0$, $b_{22} > 0$ and $b_{12} b_{21} < 0$, the critical parameter β_{**}, found on the assumption that slight damping occurs but gradually vanishes, is smaller in modulo than or equal to (for $\eta = 1$) the critical value β_* for a system without damping.

CHAPTER 2

STABILITY OF EQUILIBRIUM OF ELASTIC SYSTEMS IN THE PRESENCE OF FOLLOWER FORCES

2.1. HISTORICAL BACKGROUND

The problem of the stability of an elastic system under the action of follower forces was, it appears, first investigated by E. L. Nikolai. In one of his papers first published in 1928[1] he investigated the stability of the rectilinear form of a flexible bar, one end of which is fully fixed while the other end (which is free) is subjected to a compressive force and a twisting moment. He found that when the vector of the moment is "tangential" (i.e. is directed along the tangent to the deformed axis of the bar), no forms of equilibrium can exist other than the rectilinear form. Nikolai concluded that the usual method of determining the critical force in this problem is invalid. After deriving the equation for small oscillations of the bar about its rectilinear form of equilibrium, Nikolai was able to show that this equation is unstable no matter what the value of the torque (provided damping is ignored and a bar of circular section is considered). In his next paper[2] it was shown that if the bar has different bending stiffnesses the rectilinear form of the bar is stable at sufficiently low values of the torque. Furthermore, there is a critical value of the torque at which the rectilinear form ceases to be stable.

In 1930, at the International Congress of Applied Mechanics, E. L. Nikolai gave a paper on the stability of flexible shafts.[3] In this paper he summarized the results of his researches into the stability of bars in compression and torsion, and extended them to the case of rotating shafts. Subsequently, Nikolai's results were extended in a paper by Dzhanelidze[1], who, by analyzing the exact equations of bending for a bar, was able to show that in the problem investigated by Nikolai there are definitely no forms of equilibrium other than the rectilinear form. Different types of support conditions have been investigated by Shashkov.[2]

---

[1] NIKOLAI, E. L. On the stability of the rectilinear form of equilibrium of a bar in compression and torsion. *Izv. Leningr. politekhn. in-ta*, **31** (1928); see also NIKOLAI, E. L. *Studies in Mechanics* (Trudy po mekhanike) Gostekhizdat, 1955.

[2] NIKOLAI, E. L. On the problem of the stability of a bar in torsion, *Vestn. prikl. matem. mekh.*, **1** (1929).

[3] NIKOLAI, E. L. Über den Einfluß der Torsion auf die Stabilität rotierender Wellen, *Proc. of the 3rd Congr. Appl. Mech.*, Stockholm 1930.

Until recently, there were many aspects of Nikolai's problem that were unclear, and this gave rise to what was known as "Nikolai's paradox". In recent years there has been a marked growth of interest in problems of the stability of shafts in torsion. In 1951 Morris[3] once more considered the problem of a cantilever bar of circular section under the action of an axial force and tangential twisting moment, having in mind its application to shafts in aircraft gas turbines. The same year saw the publication of the first of a series of investigations by Ziegler[4], who considered the problem of the stability of a flexible bar under various conditions of torque, and confirmed many of the results obtained by Nikolai. Ziegler classified the various cases of torque and singled out those that corresponded to external forces having a potential. In particular, he showed that the moment whose vector bisects the angle between the tangent to the undeformed and deformed axes of the bar is conservative. (The fact that in this case the problem of stability in the Euler sense has a solution was discovered by E. L. Nikolai in 1928.) Further investigations of these problems were carried out by Troesch.[5]

[1] Dzhanelidze, G. Iu. On the question of the form of equilibrium of a bar in compression and torsion. *Tr. Leningr. industr. in-ta*, No. 3, 1st Edition, 1939.

[2] Shashkov, I. E. On the stability of the rectilinear form of equilibrium of a drilling shaft. *Inzh. sborn.* **1**, No. 1 (1941). On the stability of a prismatic bar of arbitrary cross-section in compression and torsion. *Inzh. sborn.* 7, 1950.

[3] Morris, J. Torque and the flexural stability of a cantilever, *Aircraft Eng.* **23**, No. 274 (1951).

[4] Ziegler, H. Ein nichtkonservatives Stabilitätsproblem. *Z. angew. Math. Mech.* **31**, Nos. 8/9 (1951); Stabilitätsprobleme bei geraden Stäben und Wellen. *Z. angew. Math. Phys.* **2**, No. 4 (1951); Knickung gerader Stäbe unter Torsion, *Z. angew. Math. Phys.* **3**, No. 2 (1952); Kritische Drehzahlen unter Torsion und Druck, *Ing. Arch.* **20**, No. 6 (1952).

[5] Troesch, A. Stabilitätsprobleme bei tordierten Stäben und Wellen, *Ing.-Arch.* **20**, No. 4 (1952).

Supplementary information can be obtained from the surveys by Ziegler.[1]

In 1939, Reut[2] formulated another non-conservative problem of elastic stability. He considered a bar clamped at one end and subjected at the other to a compressive force the line of action of which remains constant as the bar deforms. This type of loading can be produced, for example, by the pressure of a jet of liquid or gas the inclination of which as the rod deforms can be ignored. As in the problem of E. L. Nikolai, it was shown that forms of equilibrium, other than the rectilinear form, do not exist. B. L. Nikolai[3] considered small oscillations of the bar about its undeformed position, and in this way found the critical value of the compressive force.

The next series of problems to attract considerable interest was concerned with the stability of a flexible bar fully fixed at one end and subjected at the other to a tangential compressive force. It was shown by Feodos'ev[4] and Pflüger[5] that in this problem there are no forms of equilibrium close to the undeformed form. The critical force for this problem was calculated by Beck[6], as well as by Deineko and Leonov.[7] In the latter reference an approximate solution also was given for a bar with two concentrated masses attached. The case of a mass concentrated at the end of the bar was analyzed by Dzhanelidze[8], who assumed the combined effect of a tangential force and a force acting in a constant direction. The case of a distributed and a concentrated mass acting together was considered by

---

1. Ziegler, H. Linear elastic stability, *Z. angew. Math. Phys.* 4, Nos. 2—3 (1953); see also Advances in Applied mechanics, Vol. 4, Academic Press, N. Y. 1956.

2. Reut, V. I. On the theory of elastic stability. *Tr. Odessk. in-ta inzh. grazhd. i komm. str-va*, No. 1 (1939).

3. Nikolai, B. L. On the stability criterion of elastic systems. *Tr. Odessk. in-ta inzh. grazhd. i komm. str-va*, No. 1 (1939).

4. Feodos'ev, V. I. Selected Problems and Questions in Strength of Materials (*Izbrannye zadachi i voprosy po soprotivleniiu materialov*) pp. 38 and 165, Gostekhizdat, 1953.

5. Pflüger, A. *Stabilitätsprobleme der Elastostatik*, p. 217, Springer-Verlag, Berlin 1950.

6. Beck, M. Die Knicklast des einseitig eingespannten tangential gedrückten Stabes. *Z. angew. Math. Phys.* 3, No. 3 (1952).

7. Deineko, K. S. and Leonov, M. Ia. The dynamic method of investigating the stability of a bar in compression. *PMM*, **19**, No. 6 (1955).

8. Dzhanelidze, G. Iu. On the stability of a rod under the action of a follower force. *Tr. Leningr. politekhn. in-ta*, No. 192 (1958).

Pflüger[1]. The paper by Kopeikin and Leonov[2] also belongs to this group of problems.

Feodos'ev[3] pointed out a further problem in which there is no form of equilibrium close to the undisturbed form. This was the problem of the stability of the plane form of bending of a narrow strip under the action of a moment of constant magnitude and direction. So far the critical value of the moment for this case has not been found.[4] The author[5] considered another problem of the stability of the plane form of bending. This problem proved to be a very convenient model for demonstrating the various phenomena inherent in non-conservative problems of the theory of dynamic stability and was widely used in another paper by the author.[6]

There are two further problems of elastic stability in the presence of follower forces that should be mentioned. The author[7] considered the problem of the stability of the symmetrical form of bending of a circular arc under the action of a concentrated force which rotates together with the section during deformation. He found that in this case loss of stability occurs by a bifurcation of the forms of equilibrium in spite of the fact that the loading is not conservative. Ziegler[8], in a paper dealing with the analysis of the static criterion of stability,

---

1. PFLÜGER, A. Zur Stabilität des tangential gedrückten Stabes. *Z. angew. Math. Mech.* 35, No. 5 (1955).

2. KOPEIKIN, Iu. D. and LEONOV, M. Ia. On a particular case of loss of stability of a bar in compression. *PMM*, **19**, No. 6 (1955). In connection with this article see "*Mekhanika*", No. 4, review No. 4535 (1958).

3. See reference 3 on p. 88.

4. Calculations for a very similar problem were recently given in an article: GOPAK, K. N. and KRIVOSHEEVA, S. G. Bending-torsional vibrations and stability of the plane form of bending of a cantilever strip. *Izv. Akad. Nauk USSR*, OTN, "*Mekhanika i mashinostroenie*", No. 4 (1959).

5. BOLOTIN, V. V. *Dynamic Stability of Elastic Systems* (Dinamicheskaia ustoichivost' uprugikh sistem). Gostekhizdat, 1956.

6. BOLOTIN, V. V. On vibrations and stability of bars under the action of non-conservative forces. *Sb.* "*Kolebaniia v turbomashinakh*", Izd-vo. Akad. Nauk USSR, 1959.

7. BOLOTIN, V. V. On parametrically excited vibrations of elastic arcs. *Dokl. Akad. Nauk USSR*, **88**, No. 4 (1952).

8. ZIEGLER, H. Die Stabilitätskriterien der Elastomechanik, *Ing.-Arch.* **20**, No. 1 (1952). Non-conservative stability problems of bars were considered by BARTA [BARTA, J., Becksches Stabilitätsproblem und verwandte Probleme, *Acta techn. Acad. Sci. Hung.*, **31**, No. 1—2 (1960)] and LEIPHOLZ [LEIPHOLZ, H., Anwendung des Galerkinschen Verfahrens auf nichtkonservative Stabilitätsprobleme des elastischen Stabes, *Z. angew. Math. Phys.*, **8**, No. 4 (1962)].

used as a model the problem of the oscillations of a double pendulum coupled elastically and subjected to a tangential force.

It what follows we shall give results for some of the problems listed above. In addition, we shall also consider some new problems.

2.2. PROBLEM OF THE STABILITY OF A BAR COMPRESSED BY A TANGENTIAL FORCE

Let us consider a slender elastic bar of constant cross-section under the action of a tangential force and performing small oscillations about its undisturbed (rectilinear) form of equilibrium. We shall confine our attention to the case of oscillations which take place in one of the principal planes of inertia (Fig. 35).

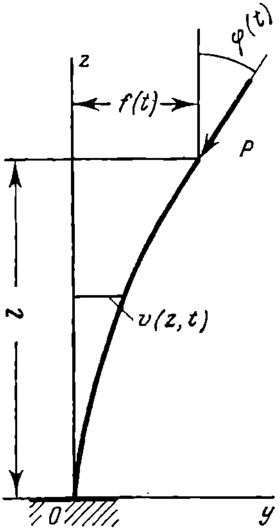


Fig. 35

Let $v(z, t)$ be the small deflection at any point on the bar, $f(t)$ the deflection at its free end, $\varphi(t)$ the angle of rotation of its end section and EJ the bending stiffness of its section. Within the framework of the usual assumptions of the elementary theory of bending, the equation of small oscillations of the bar is of the form

$$EJ\frac{\partial^2 v}{\partial z^2} = P(f - v) - P\varphi(l - z) + L_J,$$

where L_J is the bending moment produced by the action of inertia forces (in the sense of d'Alembert's principle). Differentiating this equation twice and noting that

$$\frac{\partial^2 L_J}{\partial z^2} = - m \frac{\partial^2 v}{\partial t^2},$$

where m is the mass of the bar per unit length, we obtain

$$EJ \frac{\partial^4 v}{\partial z^4} + P \frac{\partial^2 v}{\partial z^2} + m \frac{\partial^2 v}{\partial t^2} = 0. \tag{2.1}$$

Equation (2.1) is exactly the same as the corresponding equation for the case of a conservative force acting in a fixed direction. The conditions at the fixed end are also the same:

$$v(0, t) = \frac{\partial v(0, t)}{\partial z} = 0. \tag{2.2}$$

The conditions at the free end can be written in the form

$$\frac{\partial^2 v(l, t)}{\partial z^2} = \frac{\partial^3 v(l, t)}{\partial z^3} = 0 \tag{2.3}$$

in place of the conditions

$$\frac{\partial^2 v(l, t)}{\partial z^2} = 0, \qquad \frac{\partial^3 v(l, t)}{\partial z^3} = - \frac{P}{EJ} \frac{\partial v(l, t)}{\partial z} \tag{2.4}$$

in the case of a conservative force.

Equation (2.1) will be satisfied if we set

$$v(z, t) = V(z)\, e^{i\Omega t}, \tag{2.5}$$

where $V(z)$ is an unknown function and Ω is an unknown and, in general, complex constant. Substitution into Equation (2.1) gives

$$\frac{d^4 V}{d\zeta^4} + \beta \frac{d^2 V}{d\zeta^2} - \omega^2 V = 0, \tag{2.6}$$

where we have introduced the non-dimensional parameters

$$\zeta = \frac{z}{l}, \qquad \beta = \frac{P l^2}{EJ}, \qquad \omega = \Omega l^2 \sqrt{\frac{m}{EJ}}. \tag{2.7}$$

The general solution to Equation (2.6) is of the form

$$V(\zeta) = C_1 \sin r_1 \zeta + C_2 \cos r_1 \zeta + C_3 \sinh r_2 \zeta + C_4 \cosh r_2 \zeta,$$

where

$$\left.\begin{aligned} r_1^2 &= \frac{\beta}{2} + \sqrt{\left[\left(\frac{\beta}{2}\right)^2 + \omega^2\right]}, \\ r_2^2 &= -\frac{\beta}{2} + \sqrt{\left[\left(\frac{\beta}{2}\right)^2 + \omega^2\right]}. \end{aligned}\right\} \tag{2.8}$$

If we now satisfy the boundary Conditions (2.2) and (2.3) we obtain the following equations for finding the constants of integration:

$$C_2 + C_4 = 0, \qquad r_1 C_1 + r_2 C_3 = 0,$$

$$C_1(r_1^2 \sin r_1 + r_1 r_2 \sinh r_2) + C_2(r_1^2 \cos r_1 + r_2^2 \cosh r_2) = 0,$$

$$-C_1(r_1^3 \cos r_1 + r_1 r_2^2 \cosh r_2) + C_2(r_1^3 \sin r_1 - r_2^3 \sinh r_3) = 0.$$

Equating the determinant to zero, we obtain the characteristic equation

$$\begin{vmatrix} r_1 \sin r_1 + r_2 \sinh r_2 & r_1^2 \cos r_1 + r_2^2 \cosh r_2 \\ -r_1^2 \cos r_1 - r_2^2 \cosh r_2 & r_1^3 \sin r_1 - r_2^3 \sinh r_2 \end{vmatrix} = 0$$

or

$$\Delta(r_1, r_2) = r_1^4 + r_2^4 + r_1 r_2 (r_1^2 - r_2^2) \sin r_1 \sinh r_2 + \\ + 2 r_1^2 r_2^2 \cos r_1 \cosh r_2 = 0.$$

Returning now to the definitions (2.7), we obtain

$$\Delta(\omega, \beta) = \beta^2 + 2\omega^2 + \beta\,\omega \sin r_1 \sinh r_2 + 2\omega^2 \cos r_1 \cosh r_2 = 0. \tag{2.9}$$

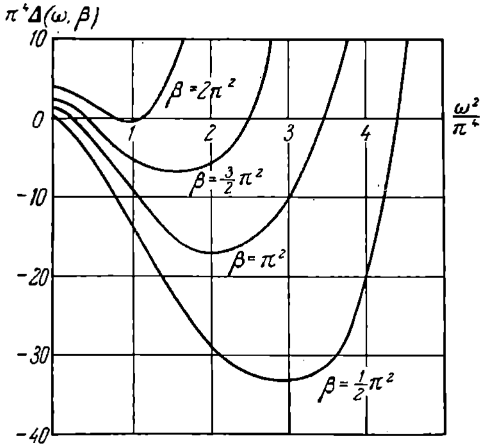


Fig. 36

If there is no external force, $\beta = 0$ and the roots of Equation (2.9) are real. They correspond to the frequencies of natural oscillations of the unloaded bar. As the parameter β increases the two smallest roots move closer, and at some value of $\beta = \beta_*$ become multiple (Fig. 36). With further increase in β the roots become complex, one of them having a negative imaginary part. Consequently, as follows from Formula (2.5), the value β_* corresponds to the critical force P_*. According to Beck's calculations this value is $\beta_* = 20.05$, and according to Deineko and Leonov, $\beta_* = 2.0028\pi^2 = 19.77$. Approximately, we can set

$$P_* \approx \frac{2\pi^2 EJ}{l^2}. \tag{2.10}$$

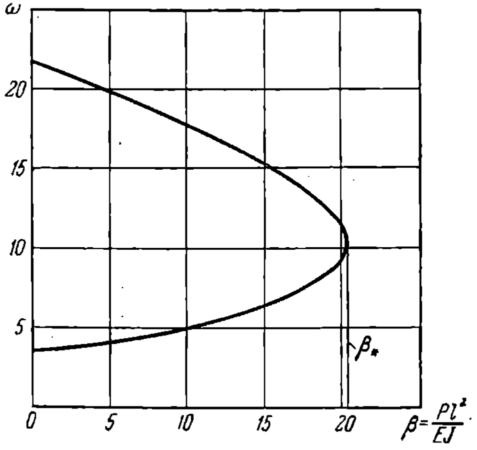


Fig. 37

The graph of the variation of the frequencies ω in the region of real values is shown in Fig. 37.

2.3. Influence of Mass Distribution

Pflüger[1] considered the case when, in addition to the distributed mass, the bar has a concentrated mass M at the end (Fig. 38). Equation (2.1), the boundary conditions (2.2) and the first of Conditions

---

1. See reference 5 on page 88.
2. We have used the results of their calculations in the introduction (Section 6).

(2.3) are unaltered. Instead of the fourth condition we must set

$$\frac{\partial^2 v(l,t)}{\partial z^2} = \frac{M}{EJ}\,\frac{\partial^2 v(l,t)}{\partial t^2}. \tag{2.11}$$

Repeating the procedure of the preceding Section and applying boundary Condition (2.11), we arrive at an equation which is a generalization of Equation (2.9):

$$\Delta(\omega,\beta) = \beta^2 + 2\omega^2 + \beta\,\omega \sin r_1 \sinh r_2 + 2\omega^2 \cos r_1 \cosh r_2 -$$

$$- \frac{M}{m\,l}\,2\omega \sqrt{\left[\left(\frac{\beta}{2}\right)^2 + \omega^2\right]}\,(r_2 \sin r_1 \cosh r_2 - r_1 \cos r_1 \sinh r_2).$$

The results of a numerical solution of this equation are plotted in Fig. 39. If $M = 0$, then $\beta_* \approx 20.05$. If the mass of the bar is negligibly small compared with the concentrated mass M at the end, then β is equal to the smallest root of the equation $\tan \sqrt{\beta} = \sqrt{\beta}$, i.e. $\beta_* = 20.19$.

The case of two concentrated masses which are large compared with the mass of the bar was investigated by Deineko and Leonov.[2]

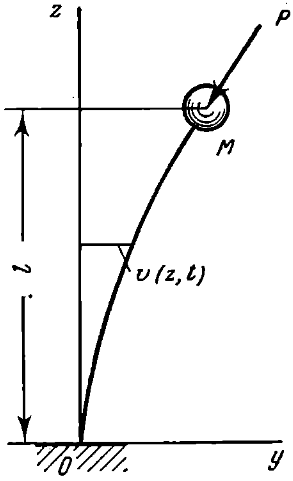


Fig. 38

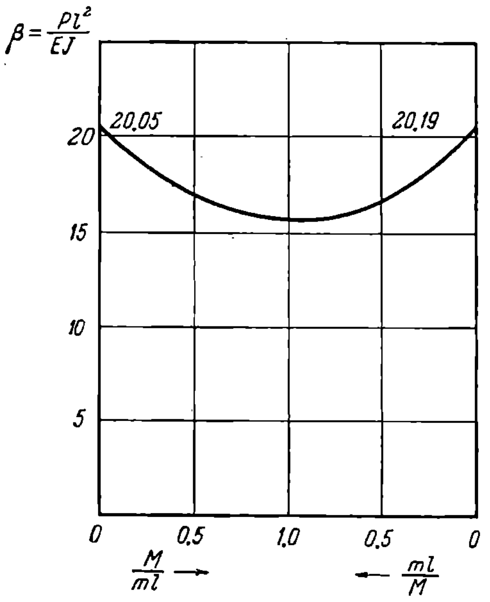


Fig. 39

2.4. Approximate Solution of the Problem

An exact investigation of the stability of bars under the action of non-conservative forces entails considerable computational difficulties, associated with the behavior of the roots of transcendental equations in the plane of the complex variable. These difficulties become even more severe if the problem includes any complications (for instance, variable sections, complex loading, damping effects etc.). In cases such as these the approximate method outlined in Section 1.14 will undoubtedly have many advantages.

In the case of a bar under the action of tangential forces Equations (1.79) assume the form

$$\frac{d^2 f_j}{dt^2} + \varepsilon_j \frac{df_j}{dt} + \Omega_j^2 f_j + \Omega_j^2 \beta \sum_{k=1}^{\infty} b_{jk} f_k = 0 \quad (j = 1, 2, \ldots), \tag{2.12}$$

where $f_j(t)$ are the coefficients of the series expansion of the deflection $v(z, t)$ in the modes of natural small oscillations of the bar $v_j(z)$, Ω_j are the frequencies of these oscillations, ε_j are the damping coefficients, β is a parameter to the accuracy of which the values of the longitudinal force $N(z)$ at each section of the bar are given, and b_{jk} are coefficients given by Formula (1.71):

$$b_{jk} = -\frac{1}{\Omega_j^2} \int_0^1 N(z)\, v_j \frac{d^2 v_k}{dz^2}\, dz. \tag{2.13}$$

Formula (2.13) is given for the case when the functions $v_j(z)$ satisfy the normalizing condition

$$c_j = \int_0^1 m(z)\, v_j^2\, dz = 1.$$

If this condition is not satisfied, then instead of Formula (2.13), we obtain

$$b_{jk} = -\frac{1}{\Omega_j^2 \sqrt{(c_j c_k)}} \int_0^{\ell} N(z)\, v_j \frac{d^2 v_k}{dz^2}\, dz. \tag{2.14}$$

Formula (2.13) was obtained in Chapter I from the general equations of the theory of elastic stability; we will show briefly how this formula can be obtained directly from the equation of bending oscillations of the bar. If all the longitudinal forces acting on the bar are tangential, it can easily be shown that this equation is of the form

$$\mathfrak{L}(v) = \frac{\partial^2}{\partial z^2}\left[E\,J(z)\frac{\partial^2 v}{\partial z^2}\right] + m\frac{\partial^2 v}{\partial t^2} - \beta\,N(z)\frac{\partial^2 v}{\partial z^2} = 0. \quad (2.15)$$

The boundary conditions depend on the type of support; the tangential forces, however, do not enter in the dynamic boundary conditions.

We shall represent the solution to Equation (2.15) in the form of a series

$$v(z, t) = \sum_{j=1}^{\infty} f_j(t)\, v_j(z)$$

and apply Galerkin's variational method. Taking into account that by virtue of the remark concerning the boundary conditions, the equations of Galerkin's method for this problem will contain no terms outside the integral signs, i.e. they will be of the form

$$\int_0^l \mathfrak{L}(v)\, v_j\, dz = 0 \qquad (j = 1, 2, \ldots),$$

we easily obtain Equations (2.12).

We shall use Equations (2.12) to solve approximately the problem of Section 2.2. For this problem $N(z) = -P$,

$$v_j(\zeta) = \sin\lambda_j\zeta - \sinh\lambda_j\zeta - \mu_j(\cos\lambda_j\zeta - \cosh\lambda_j\zeta),$$

$$\Omega_j = \frac{\lambda_j^2}{l^2}\sqrt{\frac{E\,J}{m}}, \qquad \zeta = \frac{z}{l}.$$

Here λ_j and μ_j are constants. For example, $\lambda_1 = 1.875$, $\lambda_2 = 4.694$, $\mu_1 = 1.3622$, $\mu_2 = 0.98187$. For the parameter β, which is a characteristic of the external loading, it is convenient, as before, to take the quantity

$$\beta = \frac{P\,l^2}{E\,J}. \quad (2.16)$$

The matrix elements b_{jk} then become non-dimensional. Setting

$$f_j = F_j\, e^{i\Omega t}, \quad (2.17)$$

we reduce the problem to an investigation of the roots of a characteristic equation of the type (1.82)

$$|(\omega_j^2 - \omega^2)\,\delta_{jk} + \beta\, b_{jk}| = 0, \quad (2.18)$$

where $\omega_j = \Omega_j/\Omega_1$, $\omega = \Omega/\Omega_1$.

For small values of β all the non-dimensional frequencies ω are real; as before we classify this case as stability of the undisturbed (rectilinear) state of the bar. An approximate analysis of stability reduces to the determination of conditions for which the truncated equations obtained from the infinite determinant (2.18) have no roots lying in the right-hand half-planes of the complex variable.

As a first approximation we shall take a second order determinant, and use the results of Section 1.16. In this case $\omega_2 = 6.250$, $b_{11} = 0.0666$, $b_{12} = -0.954$, $b_{21} = 0.00387$, $b_{22} = -0.0274$. The conditions $a_2(\beta) > 0$, $a_4(\beta) > 0$, where a_2 and a_4 are given by (1.103), are satisfied in this problem for all values of β. After substitution of the expressions for $a_2(\beta)$ and $a_4(\beta)$ and the numerical values, Condition (1.104) becomes

$$\beta^2 - 120.5\,\beta + 2020 > 0.$$

From this we obtain a critical value of the parameter β of $\beta_* = 20.15$, and consequently the critical force will be

$$P_* = 20.15\,\frac{EJ}{l^2}.$$

This value differs by approximately 0.5% from that found by Beck.

We can now study the behavior of the characteristic exponents $\sigma = i\,\omega$ in the plane of the complex variable with gradual increase in the external load (Fig. 40). If $\beta < \beta_* = 20.15$ all the exponents remain on the imaginary axis. At $\beta = \beta_*$ the characteristic exponents become multiple in pairs, and with further increase in β we have a case of typical oscillatory instability. This lasts as long as $\beta < \beta_*$

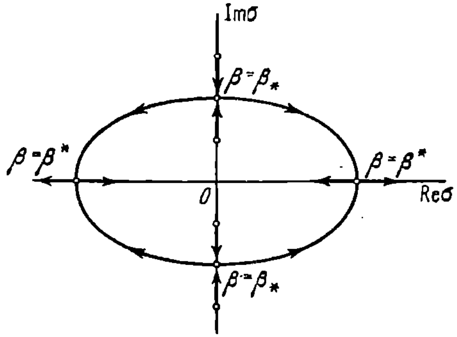


Fig. 40

= 100.35. Thereafter the characteristic exponents become purely real, and since two of them are positive, we have a case of instability with exponentially increasing disturbances.

It should be pointed out that, as the general theory shows, these results would be modified by taking into account the effect of damping.

2.5. EFFECT OF DAMPING ON STABILITY

Consider now Equations (2.19), which contain the first derivatives of the generalized coordinates f_j with respect to time. In general the damping coefficients are different for different modes of oscillation and, moreover, they can depend on the frequencies of the particular motion considered. Making use of the results of Section 1.16, and assuming that there is only slight damping, we arrive at the stability condition (1.101) with the coefficients (1.102):

$$A_1(\eta)\beta^2 + B_1(\eta)\beta + C_1(\eta) > 0, \qquad (2.19)$$

where η is the ratio of the partial characteristics of damping:

$$\eta = \frac{\varepsilon_2}{\varepsilon_1}.$$

There is no need to give the explicit expressions for the coefficients of Equation (2.19). The roots of the equation, found by equating the left-hand side of Expression (2.19) to zero, are the critical parameters β_{**} for the case when damping is taken into account[1]. The graph of the parameter β_{**} for various values of the ratio of the partial characteristics of damping is shown in Fig. 41. Only when $\varepsilon_1 = \varepsilon_2$ does the equality $\beta_{**} = \beta_*$ exist. In all other cases $\beta_{**} < \beta_*$.

The most obvious measure of damping is the relative dissipation of energy during one oscillation cycle (more details of this are given in Section 3.4):

$$\psi_j = \frac{2\pi\Omega\,\varepsilon_j}{\Omega_j^2}.$$

Here Ω is the real frequency for a periodic process which occurs at the boundary of the region of stability. Thus

$$\eta = \frac{\psi_2}{\psi_1}\,\frac{\Omega_2^2}{\Omega_1^2}.$$

---

[1] More exactly, when very slight damping which gradually vanishes is taken into account.

As an example, let us consider the case when $\psi_1 = \psi_2$, which corresponds to the quite plausible assumption that the relative dissipation of energy is independent of frequency. Here $\eta = \omega_2^2 = 39.06$, and the critical parameter β_{**}, found by taking into account slight damping which then vanishes, is considerably less than the parameter β_* (Fig. 41).

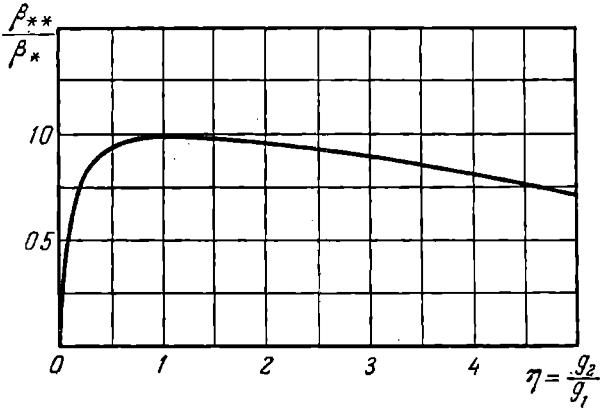


Fig. 41

The question of how the stability of non-conservative systems is affected by dissipative forces has already been discussed (in Section 1.16). Here we encounter an effect which is completely at variance with our intuitive concept of dissipative forces as a stabilizing factor. They do have a stabilizing effect provided the external forces acting on the system have a potential (this is stated in one of Kelvin's theorems). Tangential forces do not have a potential.

It is perhaps worthwhile to emphasize once more the different interpretation of the concept of stability which was employed in the two preceding Sections. In the last Section undisturbed equilibrium was considered to be stable if all the characteristic exponents were found to be on the imaginary axis. In this Section we assume that stability exists when all the characteristic exponents lie to the left of the imaginary axis. Since the present physical problem is non-linear, and since according to Liapunov the case of purely imaginary characteristic exponents is one of the doubtful cases (when the linear approximation is insufficient for an assessment of stability), there is every reason to check the solutions obtained taking damping into account. This whole problem requires further investigation, and in particular, investigation based on non-linear equations.[1]

2.6. PROBLEM OF THE STABILITY OF A BAR COMPRESSED BY A FORCE WITH A FIXED LINE OF ACTION

We shall consider briefly the case of a force with a fixed line of action, following initially the method of B. L. Nikolai. We shall neglect the mass of the bar and assume that the whole mass is concentrated at one point at the free end of the bar (Fig. 42). Let M be the magnitude of this mass and $f(t)$ the deflection at the free end. The equation for small bending oscillations can be written in the form

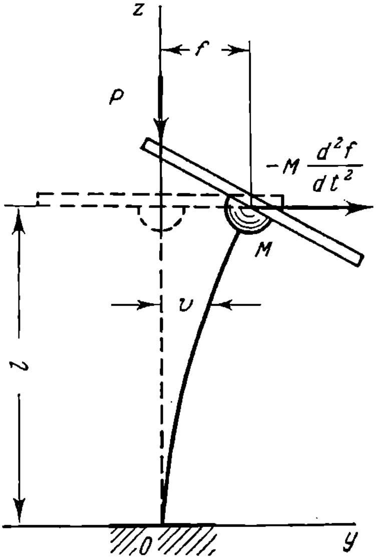


Fig. 42

$$EJ\frac{\partial^2 v}{\partial z^2} = -Pv - M(l-z)\frac{d^2f}{dt^2}. \tag{2.20}$$

---

[1] Certain interesting results are presented in the papers: ZORI'I, L. M., LEONOV, M. Ia., Influence of friction on stability of nonconservative systems, *Nauchn. zapiski In-ta mashinovedeniia i avtomatiki AN USSR*, 7, No. 7 (1961); LEONOV, M. Ia., ZORI'I, L. M., On the influence of friction on the critical loading of the compressed bar, *Dokl. Akad. Nauk SSSR*, **145**, No. 2 (1962).

As usual, we set

$$v(z, t) = V(z)\, e^{i\Omega t}; \quad f(t) = F\, e^{i\Omega t}. \tag{2.21}$$

Substitution into (2.20) gives the equation

$$\frac{d^2 V}{dz^2} + k^2 V = \frac{M \Omega^2 F}{E J}(l - z),$$

the solution of which is

$$V(z) = C_1 \sin k z + C_2 \cos k z + \frac{M \Omega^2 F}{k^2 E J}(l - z)$$

(here, as before, $k^2 = P/EJ$). If we now subject this solution to the boundary conditions

$$V(0) = \frac{d V(0)}{dz} = 0, \quad V(l) = F,$$

we readily obtain the formula for the frequency Ω:

$$\Omega = \pm \sqrt{\left[\frac{P}{M l} \frac{1}{\dfrac{\sin k l}{k l} - \cos k l}\right]}. \tag{2.22}$$

The frequences given by Formula (2.22) are real if the expression in the denominator is positive. From the condition that $\tan k l = k l$ we find that the critical force for this problem is

$$P_* = \frac{20.19\, E J}{l^2}.$$

This value, and also Formula (2.22), coincide completely with the corresponding results for the problem of the stability of a bar under the action of a tangential force (see Section 5 of the Introduction).

The analogy holds for the case of a distributed mass.[1] Equation (2.1) and the boundary conditions (2.2) at the built-in end remain unaltered, but Conditions (2.3) are replaced by

$$E J \frac{\partial^2 v(l, t)}{\partial z^2} = - P\, v(l, t); \qquad E J \frac{\partial^3 v(l, t)}{\partial z^3} = - P \frac{\partial v(l, t)}{\partial z}.$$

---

[1] This was proved by E. L. Pozniak (in an unpublished paper). The analogy is based on the fact that the boundary value problems, for the two cases considered, are adjoint.

Applying to this boundary-value problem the method described in Section 2.2, we arrive at the characteristic equation

$$r_1^4 + r_2^4 + r_1 r_2 (r_1^2 - r_2^2) \sin r_1 \sinh r_2 + 2 r_1^2 r_2^2 \cos r_1 \cosh r_2 = 0,$$

where

$$r_1^2 = \frac{\beta}{2} + \sqrt{\left[\left(\frac{\beta}{2}\right)^2 + \frac{m\Omega^2 l^4}{EJ}\right]}, \qquad r_2^2 = -\frac{\beta}{2} + \sqrt{\left[\left(\frac{\beta}{2}\right)^2 + \frac{m\Omega^2 l^4}{EJ}\right]}.$$

This equation is exactly the same as the characteristic equation (2.9).

B. L. Nikolai also considered the case of the combined effect of a non-conservative force and a force which, while remaining constant in magnitude, is displaced together with the free end of the bar (Fig. 43). The latter force is obviously a usual force with a potential, as considered in the theory of elastic stability.

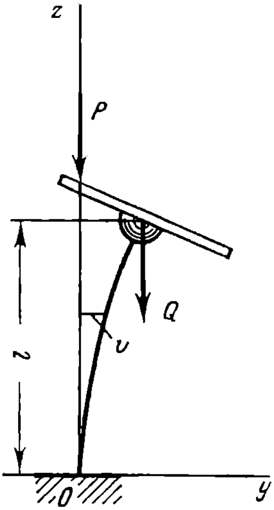


Fig. 43

The equation of small oscillations for this case is of the form

$$EJ\frac{\partial^2 v}{\partial z^2} = Pv - Q(f - v) - M(l - z)\frac{d^2 f}{dt^2}.$$

Substituting into this equation Expressions (2.21), and using the notation

$$k^2 = \frac{P + Q}{EJ},$$

we obtain

$$V(z) = C_1 \sin k z + C_2 \cos k z + \frac{[Q + M\Omega^2(l - z)]\, F}{k^2 E J}.$$

After the usual procedure we find

$$\Omega = \pm \sqrt{\left[\frac{P + Q \cos k l}{M l \left(\frac{\sin k l}{k l} - \cos k l\right)}\right]}. \tag{2.23}$$

We shall consider the following particular cases.

1. If $P = 0$, then the frequency Ω is purely imaginary, passing through the value $\Omega = 0$ when $k l = \pi/2$. This corresponds to the usual loss of stability in the Euler sense at a critical value of the compressive force given by

$$Q_* = \frac{\pi^2 E J}{4 l^2}.$$

2. If $0 < P < Q$, then loss of stability also occurs according to a type of bifurcation of the forms of equilibrium, and the critical value of the parameter $k l$ is determined from the condition

$$\cos k l = -\frac{P}{Q}.$$

It is significant that, with increase in the nonconservative component P, the critical value of the total load increases. For example, if $P = {}^1/_2 Q$ we have $k l = {}^2/_3 \pi$, i.e.

$$(P + Q)_* = \frac{4\pi^2 E J}{9 l^2}.$$

Thus the addition of a non-conservative component, provided it does not exceed a certain value, has a stabilizing effect. A similar effect is made use of in stabilizing gyroscopic systems by introducing "forces of radial correction".

3. If $P > Q$, then transition from real natural frequencies to purely imaginary frequencies through the value of $\Omega = 0$ becomes impossible. The total critical force is then given by

$$(P + Q)_* = k_*^2 E J = \frac{20.19\, E J}{l^2},$$

where k_* is the root of the equation $\tan k l = k l$.

The relation between the parameter $k\,l$ and the load ratio P/Q is shown graphically in Fig. 44. An unusual feature is the discontinuity in the magnitude of the critical force at $P/Q = 1$.

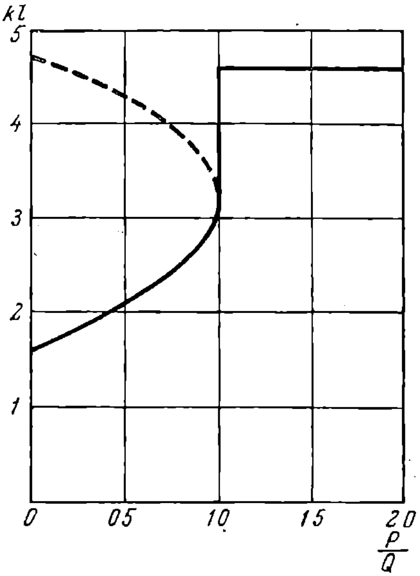


Fig. 44

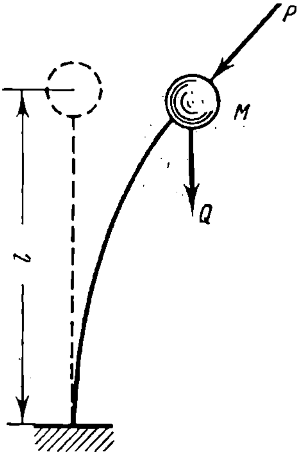


Fig. 45

Dzhanelidze[1] has shown that the problem of the stability of a bar under the action of a tangential force P and a conservative force Q (Fig. 45) also leads to formula (2.23), i.e. to the same results.

2.7. STABILITY OF THE PLANE FORM OF BENDING (DERIVATION OF THE EQUATIONS)

We proceed now to a study of the problem of the plane form of bending of bars under the action of follower forces, following for the most part the method proposed by the author[2]. Suppose that a bar with a rectilinear axis is subjected to forces perpendicular to this axis and acting in one plane. As we are not aiming at a general solution, let us assume that the plane of action of the forces is a plane of symmetry of the bar, in fact the plane of maximum stiffness (Fig. 26). The bending moment produced at various section of the bar by the external load $q(z)$ applied at a distance $e(z)$ from the axis Oz will be denoted by $L(z)$. The problem is to establish the stability conditions for the case of plane deformation of the bar, assuming that the external forces are follower forces.

---

[1] DZHANELIDZE, G. Iu. On the stability of a bar under the action of a follower force. *Tr. Leningr. politekhn. in-ta,* No. 192 (1958).

[2] BOLOTIN, V. V. On vibrations and stability of bars under the action of non-conservative forces. *Sb.* "*Kolebaniia v turbomashinakh,* Izd-vo, Akad. Nauk. USSR, 1959.

The equations of equilibrium of the bar in a position close to its original position for the case of a load which acts in a fixed direction were obtained by Vlasov[1]:

$$\left.\begin{aligned} &EJ_y\frac{d^4u}{dz^4}+\frac{d^2}{dz^2}(L_x\theta)=0,\\ &EJ_\omega\frac{d^4\theta}{dz^4}-GJ_d\frac{d^2\theta}{dz^2}-2\beta_y\frac{d}{dz}\left(L_x\frac{d\theta}{dz}\right)+\\ &\qquad+q(e-a_y)\theta+L_x\frac{d^2u}{dz^2}=0.\end{aligned}\right\} \tag{2.24}$$

Here $u(z)$ are the displacements of the bar axis from the plane of maximum stiffness, $\theta(z)$ is the angle of twist (its positive direction was indicated in Fig. 27), EJ_y, EJ_ω, GJ_d are the bending, sectorial and torsional stiffnesses of the bar, F is its area, a_y is the coordinate of the bending center and β_y is a geometrical characteristic defined by

$$\beta_y=-a_y+\frac{1}{J_x}\int\limits_F(x^2+y^2)\,y\,dF.$$

We shall list now the more common boundary conditions which will be needed later on.

I. *Hinged support.* Here the deflection, angle of rotation, as well as the bending moment and bending-twisting bimoment, are zero:

$$u=\theta=\frac{d^2u}{dz^2}=\frac{d^2\theta}{dz^2}=0.$$

---

[1] VLASOV, V. Z. *Thin-walled Elastic Bars* (Tonkostennye uprugye sterzhni) Fizmatgiz, 1959.

II. *Complete fixity.* In this case

$$u = \theta = \frac{du}{dz} = \frac{d\theta}{dz} = 0.$$

III. *Free end* (*without loading*). Here

$$\frac{d^2u}{dz^2} = \frac{d^3u}{dz^3} = \frac{d^2\theta}{dz^2} = 0, \quad \frac{d^3\theta}{dz^3} - \varkappa^2\frac{d\theta}{dz} = 0,$$

where

$$\varkappa^2 = \frac{G\,J_d}{E\,J_\omega}.$$

IV. *Free end with an applied concentrated force P.* In this case the "natural" boundary conditions (in the variational calculus sense)

$$\frac{d^2u}{dz^2} = \frac{d^2\theta}{dz^2} = 0,$$

$$E\,J_y\frac{d^3v}{dz^3} - P\,\theta = E\,J_\omega\left(\frac{d^3\theta}{dz^3} - \varkappa^2\frac{d\theta}{dz}\right) - P(e - a_y)\,\theta = 0$$

must be satisfied.

Considering the case of a follower force, we shall assume that the deviations from undisturbed equilibrium are functions of time, i.e. $u = u(z, t)$, $\theta = \theta(z, t)$. We obtain the corresponding equations by including in Equations (2.24) additional components of load which appear under conditions of bending-torsional deformation, and also the inertia terms:

$$\left.\begin{aligned} &E\,J_y\frac{\partial^4 u}{\partial z^4} + \frac{\partial^2}{\partial z^2}(L_x\theta) + m\frac{\partial^2}{\partial t^2}(u + a_y\theta) = \Delta\,q_x, \\ &E J_\omega\frac{\partial^4\theta}{\partial z^4} - G\,J_d\frac{\partial^2\theta}{\partial z^2} - 2\beta_y\frac{\partial}{\partial z}\left(L_x\frac{\partial\theta}{\partial z}\right) + q(e - a_y)\,\theta + \\ &\qquad + L_x\frac{\partial^2 u}{\partial z^2} + m\frac{\partial^2}{\partial t^2}(a_y\,u + r^2\,\theta) = \Delta\,\mu_z. \end{aligned}\right\} \quad (2.25)$$

Here m is the mass per unit length, r is the radius of gyration of the section referred to the bending center, $r^2 = a_y^2 + (J_x + J_y)/F$. Forces of inertia due to rotation and forces due to warping, which are significant only in the case of short bars in the form of shells, are ignored.

The right-hand sides of Equations (2.25) represent an additional transverse load Δq_x and an additional distributed torque $\Delta \mu_z$. In the case of forces of constant magnitude and direction $\Delta q_x = \Delta \mu_z = 0$; if the forces are follower forces, however, as will be seen from Fig. 46, we have

$$\left.\begin{aligned} \Delta q_x &= -q \sin\theta \approx -q\,\theta, \\ \Delta \mu_z &= q(e - a_y)\sin\theta \approx q(e - a_y)\,\theta. \end{aligned}\right\} \tag{2.26}$$

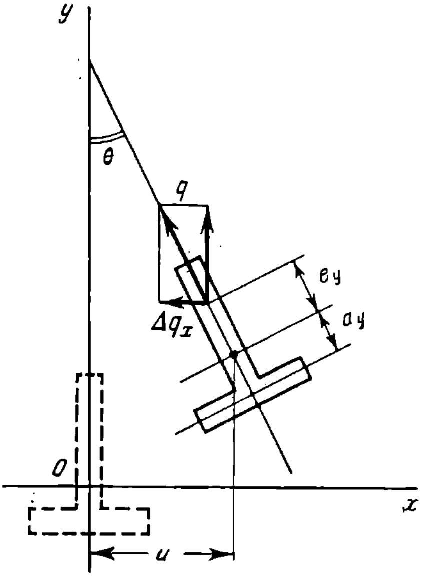


Fig. 46

By substituting (2.26) into (2.25), we obtain a system of equations

$$\left.\begin{aligned} &E J_y \frac{\partial^4 u}{\partial z^4} + \frac{\partial^2}{\partial z^2}(L_x \theta) + q\,\theta + m \frac{\partial^2}{\partial t^2}(u + a_y \theta) = 0, \\ &E J_\omega \frac{\partial^4 \theta}{\partial z^4} - G J_d \frac{\partial^2 \theta}{\partial z^2} - 2\beta_y \frac{\partial}{\partial z}\left(L_x \frac{\partial \theta}{\partial z}\right) + \\ &\qquad + L_x \frac{\partial^2 u}{\partial z^2} + m \frac{\partial^2}{\partial t^2}(a_y u + r^2 \theta) = 0. \end{aligned}\right\} \tag{2.27}$$

The boundary conditions in the case of follower forces for cases I–III are unaltered. For case IV the boundary conditions assume the form

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 \theta}{\partial z^2} = \frac{\partial^3 u}{\partial z^3} = \frac{\partial^3 \theta}{\partial z^3} - \varkappa^2 \frac{\partial \theta}{\partial z} = 0.$$

We shall attempt to reduce the system of equations in partial derivatives (2.27) to a system of ordinary differential equations. With this in mind, let us consider, in addition to Equations (2.27), the equations

$$E J_y \frac{\partial^4 u}{\partial z^4} + m \frac{\partial^2 u}{\partial t^2} = 0,$$

$$E J_\omega \frac{\partial^4 \theta}{\partial z^4} - G J_d \frac{\partial^2 \theta}{\partial z^2} - m r^2 \frac{\partial^2 \theta}{\partial t^2} = 0,$$

which describe the natural oscillations of a bar whose cross-section has two axes of symmetry. Setting $u = u_j(z) \cos \Omega_j t$, $\theta = \theta_j(z) \cos \Omega_j t$, we obtain

$$\left.\begin{aligned} E J_y \frac{d^4 u_j}{d z^4} - m \Omega_j^2 u_j = 0, \\ E J_\omega \frac{d^4 \theta_j}{d z^4} - G J_d \frac{d^2 \theta_j}{d z^2} - m r^2 \Omega_j^2 \theta_j = 0. \end{aligned}\right\} \tag{2.28}$$

Let us suppose that we have found solutions to Equations (2.28) which satisfy the boundary conditions for the original problem. These would give the modes of the natural bending oscillations $u_1(z)$, $u_2(z)$, $\ldots, u_j(z), \ldots$ and of the natural torsional oscillations $\theta_1(z), \theta_2(z), \ldots, \theta_j(z), \ldots$, which we know possess the property of orthogonality. We shall also assume them to be normalized:

$$\int_0^l m\, u_j\, u_k\, dz = \delta_{jk}, \quad \int_0^l m\, r^2\, \theta_j\, \theta_k\, dz = \delta_{jk} \tag{2.29}$$

(where δ_{jk} is the Kronecker delta).

We shall try to find a solution to (2.27) in the form

$$u(z, t) = \sum_{k=1}^{\infty} U_k(t)\, u_k(z), \quad \theta(z, t) = \sum_{k=1}^{\infty} \Theta_k(t)\, \theta_k(z), \tag{2.30}$$

using Galerkin's variational method. Since the systems of functions $u_k(z)$ and $\theta_k(z)$ are complete, the series in (2.30) are convergent. From now on, however, we shall assume that the number of terms of each series is finite and is equal to n. It should be noted that in the case of follower forces, in contrast to the case of forces of fixed direction, the equations of Galerkin's method contain no terms which take into account the "natural" boundary conditions. If we multiply the first series of (2.30) by $u_j(z)$, the second by $\theta_j(z)$, integrate over the full length of the bar and then substitute the results into Equations (2.27), we obtain the system of equations

$$\left.\begin{aligned}
&\frac{d^2 U_j}{dt^2} + \Omega_{xj}^2 U_j + \sum_{k=1}^{n} \frac{d^2 \Theta_k}{dt^2} \int_0^l m\, a_y\, u_j\, \theta_k\, dz + \\
&\qquad + \sum_{k=1}^{n} \Theta_k \int_0^l u_j \left[\frac{d^2}{dz^2}(L_x \theta_k) + q\theta_k\right] dz = 0, \\
&\frac{d^2 \Theta_j}{dt^2} + \Omega_{\theta j}^2 \Theta_j + \sum_{k=1}^{n} \frac{d^2 U_k}{dt^2} \int_0^l m\, a_y\, \theta_j\, u_k\, dz - \\
&\qquad - \sum_{k=1}^{n} \Theta_k \int_0^l 2\beta_y\, \theta_j \frac{d}{dz}\left(L_x \frac{d\theta_k}{dz}\right) dz + \\
&\qquad + \sum_{k=1}^{n} U_k \int_0^l \theta_j L_x \frac{d^2 u_k}{dz^2}\, dz = 0 \\
&\qquad\qquad (j = 1, 2, \ldots).
\end{aligned}\right\} \qquad (2.31)$$

In these equations Ω_{xj} and $\Omega_{\theta j}$ are the partial frequencies of natural bending and torsional oscillations determined as the eigenvalues corresponding to Equations (2.28) and to the boundary conditions for the bar.

As in Section (1.13), we introduce a system of continuous numeration of the generalized coordinates: $U_j = f_{2j-1}$, $\Theta_l = f_{2j}$. Then $\Omega_{xj} = \Omega_{2j-1}$, $\Omega_{\theta j} = \Omega_{2j}$, $u_j = \varphi_{2j-1}$, $\theta_j = \varphi_{2j}$. The system of Equations (2.31) can be written in the form

$$\frac{d^2 f_j}{dt^2} + \sum_{k=1}^{n} e_{jk} f_k + \Omega_j^2 \left(f_j + \beta \sum_{k=1}^{n} b_{jk} f_k\right) = 0 \qquad (2.32)$$

$$(j = 1, 2, \ldots),$$

where

$$e_{jk} = \begin{cases} \displaystyle\int_0^1 m\, a_y\, \varphi_j\, \varphi_k\, dz, & \text{if } j \pm k \text{ is an odd number,} \\ 0, & \text{if } j \pm k \text{ is an even number,} \end{cases}$$

$$b_{jk} = \begin{cases} 0, & \text{if } j \text{ and } k \text{ are odd numbers,} \\ \dfrac{1}{\Omega_j^2}\displaystyle\int_0^l \varphi_j \left[\frac{d^2}{dz^2}(L_x\,\varphi_k) + q\,\varphi_k\right] dz, & \text{if } j \text{ is an odd number, } k \text{ an even number,} \\ \dfrac{1}{\Omega_j^2}\displaystyle\int_0^l \varphi_j\, L_x \frac{d^2\varphi_k}{dz^2}\, dz, & \text{if } j \text{ is an even number, } k \text{ an odd number,} \\ -\dfrac{1}{\Omega_j^2}\displaystyle\int_0^l 2\beta_y\,\varphi_j \frac{d}{dz}\left(L_x \frac{d\varphi_k}{dz}\right) dz, & \text{if } j \text{ and } k \text{ are even numbers.} \end{cases}$$

It is assumed that the follower force is given to the accuracy of the parameter β.

Suppose that the cross-section of the bar has two axes of symmetry. Then $a_y = \beta_y = 0$, and consequently all $e_{jk} = 0$. Carrying out the integration by parts

$$\int_0^l \varphi_j \left[\frac{d^2}{dz^2}(L_x\,\varphi_k)\right] dz = \varphi_j \frac{d}{dz}(L_x\,\varphi_k)\Big|_0^l - \int_0^l \frac{d\varphi_j}{dz}\,\frac{d}{dz}(L_x\,\varphi_k)\, dz,$$

$$\int_0^l \varphi_j\, L_x \frac{d^2\varphi_k}{dz^2}\, dz = \frac{d\varphi_j}{dz} L_x\varphi_k\Big|_0^l - \int_0^l \frac{d\varphi_j}{dz}\,\frac{d}{dz}(L_x\,\varphi_k)\, dz,$$

we arrive at the formulae

$$b_{jk} = \begin{cases} 0, \\ \text{if } j \text{ and } k \text{ are either both even or both odd.} \\ \dfrac{1}{\Omega_j^2}\left[\varphi_j \dfrac{d}{dz}(L_x\,\varphi_k)\Big|_0^l - \displaystyle\int_0^l \frac{d\varphi_j}{dz}\,\frac{d}{dz}(L_x\,\varphi_k)\, dz + \int_0^l q\,\varphi_j\,\varphi_k)\, dz\right], \\ \text{if } j \text{ is an odd number and } k \text{ an even number} \\ \dfrac{1}{\Omega_j^2}\left[\dfrac{d\varphi_j}{dz} L_x\,\varphi_k\Big|_0^l - \displaystyle\int_0^l \frac{d\varphi_j}{dz}\,\frac{d}{dz}(L_x\,\varphi_k)\, dz\right], \\ \text{if } j \text{ is an even number and } k \text{ an odd number.} \end{cases} \qquad (2.33)$$

Let us compare Formulae (2.33) with Formulae (1.78), obtained from the general equations of the theory of elasticity. It is not difficult to see that in the case of boundary conditions I–III they coincide. If a concentrated force is applied to the free end of the bar the formulae differ in certain respects. This is explained by the fact that in Formulae (1.78) the concentrated force at the end is considered as a singularity in the distributed load q in Formulae (2.33)—in the terms not under the integral signs.

In the more general case of the combined action of conservative forces with a parameter α and follower forces with a parameter β we obtain the system of equations

$$\frac{d^2 f_j}{dt^2} + \sum_{k=1}^{n} e_{jk} f_k + \Omega_j^2 \left[f_j + \sum_{k=1}^{n} (\alpha\, a_{jk} + \beta\, b_{jk}) f_k \right] = 0. \quad (2.34)$$

Here the coefficients a_{jk} are defined by Formulae (1.77).

2.8. Some Numerical Results

Consider the case of a section with two axes of symmetry. Then $a_y = \beta_y = e_{jk} = 0$ and, as follows from Formulae (2.33), all the matrix elements a_{jk} and b_{jk} with indices which are not both even or both odd vanish. A number of results for systems whose matrices have this sort of structure were given in Chapter I. Here we shall confine our attention to the numerical calculation of critical forces for the problem of the stability of the plane form of bending of a beam supported at the ends and loaded at mid-span (Fig. 47). We shall take the force Q as a parameter of the conservative loading and the force P as a parameter of the follower load.

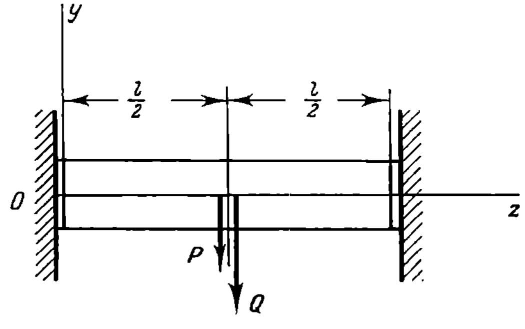


Fig. 47

We have the following expressions for the normalized modes of natural oscillations:

$$\left.\begin{aligned}\varphi_{2j-1} &= \frac{\sqrt{2}}{\sqrt{(m\,l)}}\sin\frac{j\,\pi\,z}{l},\\ \varphi_{2j} &= \frac{\sqrt{2}}{\sqrt{(m\,r^2\,l)}}\sin\frac{j\,\pi\,z}{l},\end{aligned}\right\} \tag{2.35}$$

and for the corresponding natural frequencies we have

$$\left.\begin{aligned}\Omega_{2j-1} &= \frac{j^2\,\pi^2}{l^2}\sqrt{\frac{E\,J_y}{m}},\\ \Omega_2 &= \frac{j\,\pi}{r\,l}\sqrt{\left[\frac{1}{m}\left(G\,J_d + \frac{j^2\,\pi^2}{l^2}E\,J_\omega\right)\right]}.\end{aligned}\right\} \tag{2.36}$$

As a first approximation we shall confine our attention to the basic modes of oscillations. Substituting Expressions (2.35) and (2.36) into Formulae (1.77), and taking into account that $L_x = z/2$ $(0 \leqq z \leqq l/2)$, we obtain

$$a_{12} = -\frac{1}{\Omega_1^2}\int_0^l \frac{d\varphi_1}{dz}\,\frac{d}{dz}(L_x\,\varphi_2)\,dz = -\frac{2\,l^3\,C}{\pi^4\,E\,J_y\,r},$$

$$a_{21} = -\frac{1}{\Omega_2^2}\int_0^l \frac{d\varphi_2}{dz}\,\frac{d}{dz}(L_x\,\varphi_1)\,dz = -\frac{2\,l\,r\,C}{\pi^2\left(G\,J_d + \dfrac{\pi^2}{l^2}\,E J_\omega\right)},$$

where

$$C = \frac{2\pi}{l}\int_0^{l/2}\cos\frac{\pi\,z}{l}\left(\sin\frac{\pi\,z}{l} + \frac{\pi\,z}{l}\cos\frac{\pi\,z}{l}\right)dz = \frac{1}{4}\left(1 + \frac{\pi^2}{4}\right).$$

Hence, making use of Formula (1.84), we find the critical value of the parameter Q:

$$Q_* = \frac{1}{\sqrt{(a_{12}\,a_{21})}} = \frac{\pi^3}{2C}\,\frac{1}{l^2}\sqrt{\left[E\,J_y\left(G\,J_d + \frac{\pi^2}{l^2}\,E\,J_\omega\right)\right]} \tag{2.37}$$

(this value differs by only 5% from the exact value; note that $\pi^3/2C = 17.9$).

We now evaluate the matrix elements b_{jk}. Using a delta function to express the concentrated force, we find

$$b_{12} = a_{12} + \frac{1}{\Omega_1^2}\int_0^l q\,\varphi_1\,\varphi_2\,dz = \frac{2\,l^3}{\pi^4\,E\,J_y\,r}(1 - C).$$

At the same time $b_{21} = a_{12}$. It will be found convenient to introduce the parameter

$$P_0 = \frac{1}{\sqrt{(-b_{12}b_{21})}} = \frac{\pi^3}{2\sqrt{[C(1-C)]}}\,\frac{1}{l^2}\sqrt{\left[EJ_y\left(GJ_d + \frac{\pi^2}{l^2}EJ_\omega\right)\right]}, \tag{2.38}$$

which has the dimension of a force. This parameter is related to the critical force Q_* by the expression

$$\frac{P_0}{Q_*} = \frac{\sqrt{C}}{\sqrt{(1-C)}} = 2.56. \tag{2.39}$$

The critical value of the follower load, ignoring damping, is given by Formula (1.86):

$$P_* = P_0\frac{\omega_2^2 - 1}{2\omega_2}. \tag{2.40}$$

Here ω_2 is the ratio of the partial frequencies

$$\omega_2 = \frac{\Omega_2}{\Omega_1} = \frac{l}{\pi r}\sqrt{\left(\frac{GJ_d + \frac{\pi^2}{l^2}EJ_\omega}{EJ_y}\right)}. \tag{2.41}$$

For a narrow strip of height h

$$\omega_2 \approx \frac{l}{\pi h}\sqrt{\left(\frac{24}{1+\mu}\right)},$$

where μ is Poisson's ratio. Suppose, for example, that $h = l/5$, $\mu = 0.3$. Then from Formulae (2.38) and (2.40) $P_* = 3.35\ P_0 = 8.57\,Q_*$.

It will be seen from Formula (2.40) that as the partial frequencies approach a common value, the critical value of the parameter P_* decreases. At values of the non-dimensional frequency ω_2 close to unity Formula (2.40) becomes unsuitable. A typical example of this occurs in an investigation of the stability of the plane form of bending of an I-section beam (Fig. 48). Let b be the width of the flanges, δ their thickness and h the distance between the centers of gravity of the flanges; we shall assume that the thickness of the web is negligibly small. Then $J_y \approx b^3\,\delta/6$, $J_d \approx {}^2/_3 b\,\delta^3$, $J_\omega = b^3h^2\delta/24$, $r^2 = h^2/4 + b^2/12$, whence, from Formula (2.41), we find

$$\omega_2 = \frac{1 + \dfrac{8}{\pi^2(1+\mu)}\left(\dfrac{\delta}{b}\right)^2\left(\dfrac{l}{h}\right)^2}{1 + \dfrac{b^2}{3h^2}}. \tag{2.42}$$

Suppose that $l = 400$ cm, $h = 60$ cm, $b = 20$ cm, $\delta = 1$ cm, $\mu = 0.3$. Then from Formula (2.42) $\omega_2 = 1.047$. In this case it is quite unnecessary to take into account terms containing the squares of the damping coefficients. In fact from Formula (2.40) we would find here that $P_* = 0.045\,P_0$, and from Formula (1.99)

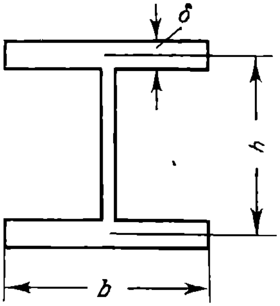


Fig. 48

$$P_{**} = \frac{P_0\sqrt{\eta}}{(1+\eta)\,\omega_2}\sqrt{\{(\omega_2^2 - 1)^2 + g^2[\eta^2 + \eta(\omega_2^2 + 1) + \omega_2^2]\}}. \tag{2.43}$$

If $\eta = 1$, $g = 0.05$, then $P_{**} = 0.0735\,P_0$, i.e. the critical force is approximately 1.6 times greater than the value found without taking damping into account.

Non-conservative forces can have a stabilizing effect if they act together with conservative forces. We can illustrate this by an example of the problem of the stability of the plane form of bending, using the results of Section (1.15). We introduce a parameter A defined by the formula

$$A = \frac{P_0Q_*}{2}(b_{12}\,a_{21} + a_{12}\,b_{21}).$$

Substituting Formulae (2.40) and (2.41), we find

$$A = \frac{2C - 1}{2\sqrt{[C(C-1)]}}. \tag{2.44}$$

Formula (2.44) gives a value of $A = 1.07$. In a manner analogous to (1.88) we introduce the parameter

$$\psi = \frac{P}{P_0} \frac{Q_*}{Q}.$$

Addition of a follower force will contribute to the stability, provided loss of stability takes place by bifurcation of the forms of equilibrium, which occurs for $\psi < \psi_0$ where

$$\psi_0 = A + \sqrt{(A^2 + 1)} = 2.53.$$

If $\psi > \psi_0$, i.e. if $P/P_0 > 2.53\, Q/Q_*$ loss of stability is of the oscillatory type.

The fact that the follower component contributes to the stability of undisturbed equilibrium when loss of stability occurs by bifurcation of the forms of equilibrium follows from physical considerations alone (Fig. 49). As the beam deflects from its position of undisturbed equilibrium the component P_x of the follower force tends to return the beam to this position.

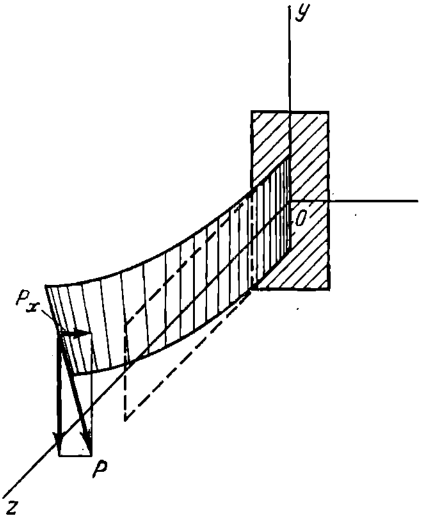


Fig. 49

2.9. Some Further Problems

We shall proceed now to examine some comparatively straightforward problems. Consider a narrow strip of rectangular section under the action of a moment L applied at the end (Fig. 50). In addition to a fixed system of coordinates $x_0\,y_0\,z_0$, we shall introduce a further system $x\,y\,z$ which rotates together with sections of the strip. Let $\boldsymbol{L}$ be the vector of the moment before deformation and $\boldsymbol{L}_1$ its vector after deformation. We shall assume that the vector $\boldsymbol{L}_1$ remains in the plane of the cross-section, but that it forms an angle $\lambda\theta$ with the direction of $\boldsymbol{L}$ proportional to the angle of rotation θ of the end section about the axis Or_0. Then, projecting the vector $\boldsymbol{L}_1$ onto the

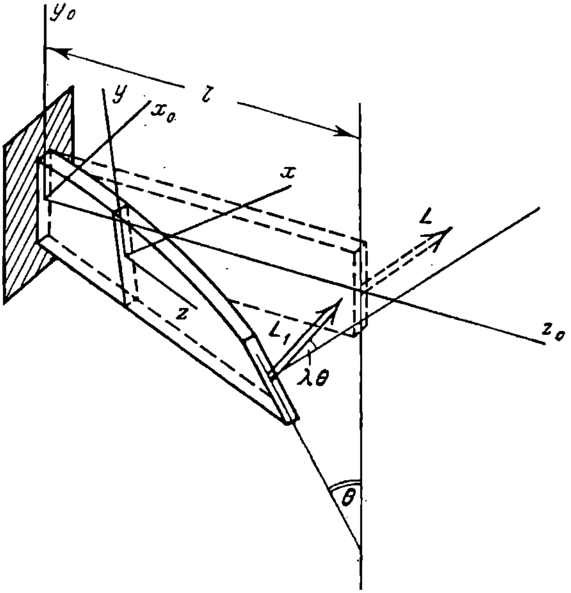


Fig. 50

fixed axes of coordinates, we find that to the accuracy of first order quantities

$$L_{x_0} = L, \quad L_{y_0} = L\lambda\,\theta(l), \quad L_{z_0} = -L\frac{du(l)}{dz}.$$

Similarly, for the moving coordinate system we find

$$L_x = L, \quad L_y = -L\theta(z) + L\lambda\,\theta(l),$$
$$L_z = L\frac{du(z)}{dz} - L\frac{du(l)}{dz}.$$

Considering an equilibrium position close to the undisturbed position, we obtain the system of equations

$$EJ_y \frac{d^2u}{dz^2} = -L\theta + L\lambda\theta(l), \qquad GJ_d \frac{d\theta}{dz} = L\frac{du}{dz} - L\frac{du(l)}{dz}$$

which are equivalent to the equation

$$\frac{d^2\theta}{dz^2} + k^2\theta = k^2\lambda\theta(l),$$

in which, by definition,

$$k^2 = \frac{L^2}{EJ_y GJ_d}.$$

The solution to this equation can be written in the form

$$\theta = C_1 \sin kz + C_2 \cos kz + \lambda\theta(l).$$

Subjecting this solution to the boundary conditions

$$\theta(0) = \frac{d\theta(l)}{dz} = 0,$$

we obtain the characteristic equation

$$\cos kl = -\frac{\lambda}{1-\lambda}.$$

This equation has real roots if $\lambda \leqq 0.5$. If this is so, loss of stability occurs by bifurcation of the forms of equilibrium. If $\lambda > 0.5$ no forms of equilibrium exist close to the undisturbed form for any value of the moment L. In this case loss of stability can only occur in the form of oscillatory instability.[1]

From now on we shall confine our attention to the case when $\lambda = 1$, the calculations for which were recently carried out by Gopak and Krivosheeva.[2] The equations of small oscillations are a particular case of Equations (2.27):

$$\left.\begin{aligned} EJ_y \frac{\partial^4 u}{\partial z^4} + L\frac{\partial^2\theta}{\partial z^2} + m\frac{\partial^2 u}{\partial t^2} &= 0, \\ -GJ_d \frac{\partial^2\theta}{\partial z^2} + L\frac{\partial^2 u}{\partial z^2} + mr^2\frac{\partial^2\theta}{\partial t^2} &= 0. \end{aligned}\right\} \tag{2.45}$$

---

[1] A similar problem was investigated by V. I. FEODOS'EV *Selected Problems and Questions in Strength of Materials* (Izbrannye zadachi i voprosy po soprotivleniiu materialov). Gostekhizdat, 1953. The only difference is that in FEODOS'EV's problem the moment vector remains in a plane parallel to *Oxy*.

[2] See reference on p. 89.

The boundary conditions are given by

$$\left.\begin{aligned} u(0) = \frac{\partial u(0)}{\partial z} = \theta(0) = 0, \\ \frac{\partial^2 u(l)}{\partial z^2} = \frac{\partial^3 u(l)}{\partial z^3} = \frac{\partial \theta(l)}{\partial z} = 0. \end{aligned}\right\} \tag{2.46}$$

As before, we shall try to find a solution to (2.45) in the form

$$u(z, t) = U(z)\, e^{i\Omega t}, \qquad \theta(z, t) = \Theta(z)\, e^{i\Omega t}.$$

For the functions $U(\zeta)$ and $\Theta(\zeta)$ ($\zeta = z/l$) we obtain the expressions

$$\begin{aligned} U(\zeta) = {} & C_1\gamma_1\cosh\mu_1\zeta + C_2\gamma_1\sinh\mu_1\zeta + C_3\gamma_2\cos\mu_2\zeta + \\ & + C_4\gamma_2\sin\mu_2\zeta + C_5\gamma_3\cos\mu_3\zeta + C_6\gamma_3\sin\mu_3\zeta, \\ \Theta(\zeta) = {} & D_1\cosh\mu_1\zeta + D_2\sinh\mu_1\zeta + D_3\cos\mu_2\zeta + D_4\sin\mu_2\zeta + \\ & + D_5\cos\mu_3\zeta + D_6\sin\mu_3\zeta. \end{aligned}$$

Here

$$\left.\begin{aligned} & \mu_{1,2} = \sqrt{\left[\mp\frac{1}{2} + \sqrt{\left(\frac{1}{4} + \omega^2\right)}\right]}, \qquad \mu_3 = ns, \\ & \gamma_1 = -\beta\pi l\sqrt{\left(\frac{GJ_d}{EJ_y}\right)}\,\frac{\mu_1^2}{\mu_1^4 - n^2} = \frac{l}{\beta\pi}\sqrt{\left(\frac{GJ_d}{EJ_y}\right)}, \\ & \gamma_2 = \beta\pi l\sqrt{\left(\frac{GJ_d}{EJ_y}\right)}\,\frac{\mu_2^2}{\mu_2^4 - n^2} = \frac{l}{\beta\pi}\sqrt{\left(\frac{GJ_d}{\pi}\right)} = \gamma_1, \\ & \gamma_3 = \beta\pi l\sqrt{\left(\frac{GJ_d}{EJ_y}\right)}\,\frac{\mu_3^2}{\mu_3^4 - n^2} = -\beta\pi l\sqrt{\left(\frac{GJ_d}{EJ_y}\right)}\,s^2, \\ & \omega^2 = \frac{m}{EJ_y}\left(\frac{l}{\beta\pi}\right)^4\Omega^2, \qquad n = \frac{m}{EJ_y}\,l^4\,\Omega^2, \\ & s = \sqrt{\left(\frac{EJ_y n^2}{GJ_d l^2}\right)}, \qquad \beta = \frac{Ml}{\pi\sqrt{(EJ_y\,GJ_d)}}. \end{aligned}\right\} \tag{2.47}$$

If we now satisfy the boundary Conditions (2.46), we obtain the characteristic equation

$$\begin{aligned} \Delta(\omega^2, \beta) = {} & \gamma_1^2\gamma_3(\gamma_1 - \gamma_3)\,\mu_1(\mu_1^2 + \mu_2^2)(\mu_3\sin\mu_3 + \mu_1\sinh\mu_1) \times \\ & \times \sinh\mu_1\cos\mu_2 - \gamma_1^2(\mu_2\sin\mu_2 + \mu_1\sinh\mu_1)\cos\mu_3[\mu_2(\gamma_1 - \gamma_3) \times \\ & \times (\gamma_3\mu_3^2 - \gamma_1\mu_2^2)\sin\mu_2 + \mu_1(\gamma_1 - \gamma_3)(\gamma_3\mu_3^2 + \gamma_1\mu_1^2)\sinh\mu_1] + \\ & + \gamma_1^2\gamma_3(\gamma_1 - \gamma_3)\,\mu_2(\mu_1^2 + \mu_2^2)(\mu_3\sin\mu_3 - \mu_2\sin\mu_2)\sin\mu_2\cosh\mu_1 - \\ & - [\gamma_1\mu_2^2(\gamma_1 - \gamma_3)\cos\mu_2 + \gamma_1\mu_1^2(\gamma_1 - \gamma_3)\cosh\mu_1] \times \\ & \times [\gamma_1(\gamma_1\mu_1^2 - \gamma_3\mu_3^2)\cos\mu_2\cos\mu_3 + \gamma_1(\gamma_3\mu_3^2 + \gamma_1\mu_1^2)\cosh\mu_1\cos\mu_3 - \\ & - \gamma_1\gamma_3(\mu_1^2 + \mu_2^2)\cosh\mu_1\cos\mu_3] = 0. \end{aligned}$$

For a sufficiently long and narrow strip this equation can be simplified. If $\delta/h \leqq 1/10$, $h/l \leqq 1/20$, then

$$s = \sqrt{\left(\frac{E J r^2}{G J_d l^2}\right)} \sim 10^{-2}$$

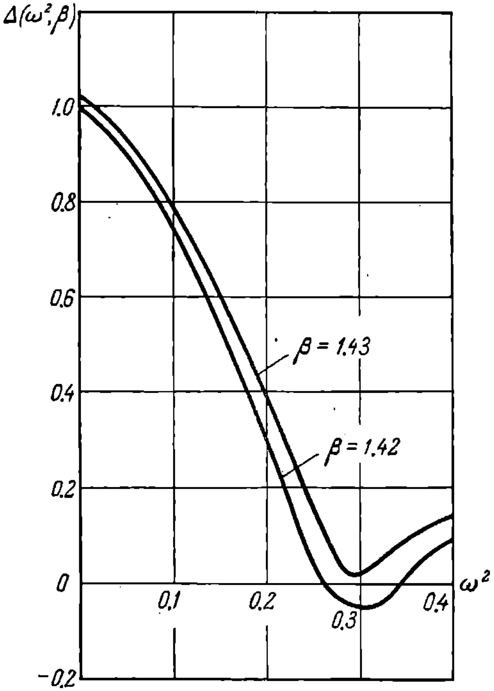


Fig. 51

and consequently $\gamma_3 \ll \gamma_1$. The calculations were carried out for the case of $E = 2.1 \times 10^6$ kg/cm², $G = 8.1 \times 10^5$ kg/cm². The results are shown in Fig. 51, from which it will be seen that the critical value of the parameter β is approximately 1.43. Recalling Definition (2.47), we find

$$M_* = 1.43 \frac{\pi \sqrt{(E J G J_d)}}{l}.$$

2.10. Equations of Equilibrium of a Bar in Compression and Torsion

The remaining sections of this chapter will be devoted to the problem of the stability of the rectilinear form of a bar in torsion for various cases of torque behavior. Proceeding along the lines suggested in the classical works of E. L. Nikolai,[1] we shall first derive the equilibrium equations for a bar in compression and torsion and in a state close to its undeformed state.

Consider a slender elastic bar of arbitrary section with its axis lying along an arbitrary curve (Fig. 52). We introduce a fixed system of coordinates $x_0\, y_0\, z_0$, the origin of which will be taken at a point A_0 on the undeformed axis. The z_0 axis will be taken as the tangent to the axis of the bar at the origin, and the axes x_0 and y_0 will be taken along the principal axes of inertia of the section. The position of the point A_0 can be specified by the arc length s_0 measured from some datum. The curvatures and twist of the undeformed bar will be denoted by p_0, q_0 and r_0.

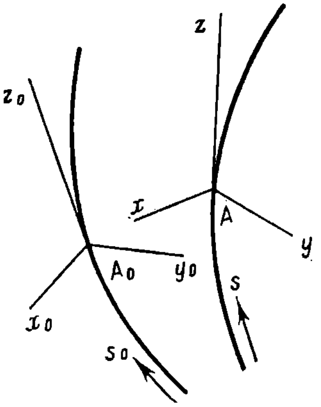


Fig. 52

Suppose that under certain conditions of loading the axis of the bar moves to a new position. The point A_0 then moves to a new position A; the components of the displacement vector of this point in the system of coordinates $x_0\, y_0\, z_0$ will be denoted by u, v and w. The coordinate system is also displaced with the bar and occupies a new position $x\, y\, z$. The angles between the components of the new coordinates axes and the corresponding original axes will be denoted by φ, ψ and χ. The values of the curvature and twist now become

---

[1] See reference on p. 86. See also the book: PONOMAREV, S. D., BIDERMAN, V. L., LIKHAREV, K. K., MAKUSHIN, V. M., MALININ, N. N., FEODOS'EV, V.I. *Strenght Analysis in Mechanical Engineering* (Raschety na prochnost' v mashinostroenii), Vol. III, Mashgiz, 1959.

$$p = p_0 + \delta p,$$
$$q = q_0 + \delta q,$$
$$r = r_0 + \delta r.$$

The nine kinematic quantities—three displacements u, v and w, three rotations φ, ψ and χ and three increments of curvatures and twist δp, δq and δr are interconnected for small strains by the six relations

$$\left.\begin{aligned}
\delta p &= \frac{d\varphi}{ds} + q_0 \chi - r_0 \psi,\\
\delta q &= \frac{d\psi}{ds} + r_0 \varphi - p_0 \chi,\\
\delta r &= \frac{d\chi}{ds} + p_0 \psi - q_0 \varphi,\\
\psi &= \frac{du}{ds} + q_0 w - r_0 v,\\
-\varphi &= \frac{dv}{ds} + r_0 u - p_0 w,\\
0 &= \frac{dw}{ds} + p_0 v - q_0 u.
\end{aligned}\right\} \qquad (2.48)$$

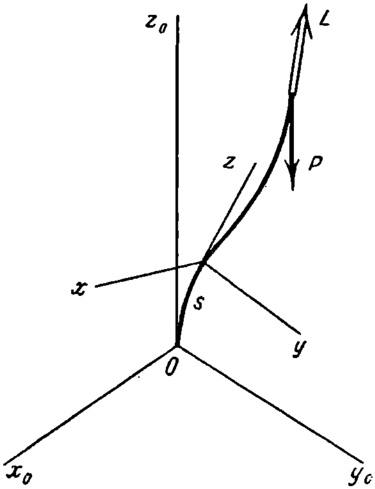


Fig. 53

The last one of these expresses the condition of inextensibility of the bar axis, namely $ds = ds_0$.

The components along the axes x, y and z of the vector sum of the internal forces on a cross-section of the bar will be denoted by V_x, V_y, and V_z, and the components of the resultant moment by L_x, L_y and L_z. Assuming that the bar is subjected to external forces of intensities f_x and f_y, and writing down the equilibrium equations for an element of the bar, we obtain the six Kirchhoff equations:

$$\left.\begin{aligned} \frac{dV_x}{ds} + q\,V_z - r\,V_y + f_x &= 0, \\ \frac{dV_y}{ds} + r\,V_x - p\,V_z + f_y &= 0, \\ \frac{dV_z}{ds} + p\,V_y - q\,V_x \qquad &= 0, \\ \frac{dL_x}{ds} + q\,L_z - r\,L_y - V_y &= 0, \\ \frac{dL_y}{ds} + r\,L_x - p\,L_z + V_x &= 0, \\ \frac{dL_z}{ds} + p\,L_y - q\,L_x \qquad &= 0. \end{aligned}\right\} \tag{2.49}$$

We obtain a closed system of equations by supplementing Equations (2.49) and the kinematic Relations (2.48) by the elastic relations

$$L_x = E\,J_x\,\delta\,p, \qquad L_y = E\,J_y\,\delta\,q, \qquad L_z = G\,J_d\,\delta\,r. \tag{2.50}$$

Here EJ_x and EJ_y are the bending stiffnesses; GJ_d is the torsional stiffness.

Making use of these equations, we can easily derive an equation which is satisfied by the deflections of the deformed bar from its rectilinear form of equilibrium (Fig. 53). Suppose that the bar is not naturally twisted. At the same time we shall assume that the twist of the rectilinear rod under the action of the external moment L can be large enough to have a significant effect on the behavior of the bar in bending (this effect is obviously the more noticeable the greater the difference between the bending stiffnesses). In order to take this finite twist into account, r_0 in Relations (2.48) should be replaced by the expression

$$r_0 = \frac{L}{G\,J_d}, \tag{2.51}$$

setting at the same time $p_0 = q_0 = 0$. This corresponds to the coordinate system $x_0\, y_0\, z_0$ being connected to the twisted bar in its rectilinear state.

In considering small deflections from the rectilinear position, we shall assume that the displacements u and v, the angles φ and ψ, and the curvatures p and q are small quantities of the first order, and we shall ignore their squares and powers of higher order. Then the third and sixth Kirchhoff equations can be written approximately as

$$\frac{dV_z}{ds} = \frac{dL_z}{ds} = 0,$$

i.e. the longitudinal force V_z and the twisting moment L_z can be considered as constants over the length of the bar. Consequently, in Equations (2.49) we can set $V_z = -P$ and $L_z = L$. Discarding from these equations quantities of the second order, we obtain a simplified system of equations

$$\left.\begin{aligned}
&\frac{dV_x}{ds} - r_0 V_y - qP + f_x = 0,\\
&\frac{dV_y}{ds} + pP + r_0 V_x + f_y = 0,\\
&\frac{dL_x}{ds} - r_0 L_y + qL - V_y = 0,\\
&\frac{dL_y}{ds} - pL + r_0 L_x + V_x = 0.
\end{aligned}\right\} \tag{2.52}$$

Eliminating V_x and V_y from these equations and using the Clebsch Relations (2.48), we obtain immediately a system of equations relating either the angles φ and ψ or the displacements u and v. The latter are particularly useful for investigating small bending oscillations about the equilibrium position. We must, however, include in the equations the inertia forces

$$f_x = -m\frac{\partial^2 u}{\partial t^2}, \qquad f_y = -m\frac{\partial^2 v}{\partial t^2}. \tag{2.53}$$

Let us consider in more detail the case of equal bending stiffnesses $EJ_x = EJ_y = EJ$. In this case all the central axes of the section are principal axes and we therefore have a wide choice in the systems of coordinates $x_0\, y_0\, z_0$ and $x\, y\, z$. In particular, we can select a system in which there is no torsion. Then in the Clebsch Relations (2.48) and the Kirchhoff Equations (2.49) we can set $r_0 = 0$

$$\left.\begin{aligned} p &= \frac{d\varphi}{ds}, & q &= \frac{d\psi}{ds}, \\ \psi &= \frac{du}{ds}, & \varphi &= -\frac{dv}{ds}, \end{aligned}\right\} \tag{2.54}$$

$$\left.\begin{aligned} &\frac{dV_x}{ds} - qP + f_x = 0, \\ &\frac{dV_y}{ds} + pP + f_y = 0, \\ &\frac{dL_x}{ds} + qL - V_y = 0, \\ &\frac{dL_y}{ds} - pL + V_x = 0, \end{aligned}\right\} \tag{2.55}$$

$$L_x = EJp, \qquad L_y = EJq. \tag{2.56}$$

2.11. STABILITY OF THE RECTILINEAR FORM OF A BAR IN COMPRESSION AND TORSION (EULER'S METHOD). CLASSIFICATION OF THE BOUNDARY CONDITIONS

Let us now consider the problem of the stability of the rectilinear form of a bar in compression and torsion when $EJ_x = EJ_y = EJ$. We shall first apply the *Euler method* to this problem. Setting $f_x = f_y = 0$ in Equations (2.55) and eliminating V_x and V_y, we obtain

$$\frac{d^2L_x}{ds^2} + \frac{dq}{ds}L + pP = 0,$$

$$\frac{d^2L_y}{ds^2} - \frac{dp}{ds}L + qP = 0.$$

Making use of Relations (2.54) and (2.56) we arrive at the system of equations

$$\left.\begin{aligned} EJu^{\mathrm{IV}} + Lv''' + Pu'' &= 0, \\ EJv^{\mathrm{IV}} - Lu''' + Pv'' &= 0. \end{aligned}\right\} \tag{2.57}$$

Here the dashes denote differentiation with respect to the arc length s, which in view of the smallness of displacements coincides with differentiation with respect to z.

The problem is to find combinations of the parameters L and P for which the boundary-value problem described by Equations (2.57) and the corresponding boundary conditions has solutions other than the trivial solution $u = v = 0$. We can speak of the eigenvalues of the boundary-value problem $L(P)$ which depend on P as a parameter. If the boundary-value problem is self-adjoint (which corresponds to the case of external forces with potential) all the eigenvalues are real, and consequently, by using Euler's method, we can find pairs of values of L and P at which bifurcation of the forms of equilibrium takes place. In this case transition from stability of the rectilinear form of equilibrium to instability occurs at certain minimum values in modulo of $L = L(P)$.

We can classify the boundary conditions for which an investigation of elastic stability by the *Euler method* leads to a self-adjoint boundary-value problem. It is wellknown that a boundary-value problem described by the differential equation $\mathfrak{L}(u) = 0$ is self-adjoint if the expression

$$\Delta = \int_0^l [u_2\,\mathfrak{L}(u_1) - u_1\,\mathfrak{L}(u_2)]\,dz \tag{2.58}$$

vanishes for any two functions $u_1(z)$ and $u_2(z)$ which satisfy the boundary conditions of the problem. The vanishing of the quantity Δ means that the reciprocal work theorem is satisfied.

Let us derive an analogous expression for the system of Equations (2.57). In this case, instead of Formula (2.58), we have

$$\Delta = \int_0^l [u_2(EJu_1^{\mathrm{IV}} + Lv_1''' + Pu_1'') - u_1(EJu_2^{\mathrm{IV}} + Lv_2''' + Pu_2'')]\,dz +$$
$$+ \int_0^l [v_2(EJv_1^{\mathrm{IV}} - Lu_1''' + Pv_1'') - v_1(EJv_2^{\mathrm{IV}} - Lu_2''' + Pv_2'')]\,dz.$$

After integrating by parts four times, we find

$$\begin{aligned}\Delta = [(EJu_1''' + Lv_1'' + Pu_1')\,u_2 + (EJv_1''' - Lu_1'' + Pv_1')\,v_2 - \\ -(EJu_1'' + Lv_1' + Pu_1)\,u_2' - (EJv_1'' - Lu_1' - Pv_1)\,v_2' + \\ +(EJu_1' + Lv_1)\,u_2'' + (EJv_1' - Lu_1)\,v_2'' - EJu_1u_2''' - EJv_1v_2''']_0^l.\end{aligned} \tag{2.59}$$

The equality $\Delta = 0$ can be satisfied in an infinite number of ways, by selecting various combinations of the eight boundary conditions for u and v. For a bar clamped at one end $u(0) = v(0) = u'(0) = v'(0) = 0$: we shall assume that these conditions are satisfied. In addition, we shall suppose that at the free end $z = l$ the conditions

$$\left.\begin{aligned} EJu''' + Lv'' + Pu' &= 0, \\ EJv''' - Lu'' + Pv' &= 0 \end{aligned}\right\} \qquad (2.60)$$

are satisfied. In order to understand the meaning of these conditions, consider the Kirchhoff equations (2.55):

$$\begin{aligned} V_x' - Pu'' &= 0, \\ V_y' - Pv'' &= 0, \\ L_x' + Lu'' - V_y &= 0, \\ L_y' + Lv'' + V_x &= 0. \end{aligned}$$

Integrating the first two equations, we obtain

$$\begin{aligned} V_x &= Pu' + \delta P_1, \\ V_y &= Pv' + \delta P_2, \end{aligned}$$

where the constants of integration δP_1 and δP_2 represent external transverse forces applied at the end of the bar. If we substitute these expressions into the other two equations we obtain

$$\left.\begin{aligned} EJu''' + Lv'' + Pu' &= -\delta P_2, \\ EJv''' - Lu'' + Pv' &= -\delta P_1. \end{aligned}\right\} \qquad (2.61)$$

From this we see that if $\delta P_1 = \delta P_2 = 0$ Conditions (2.60) are satisfied.

Taking into account Conditions (2.60), we can write Formula (2.58) in the form

$$\begin{aligned} \Delta = [-EJu_1''u_2' - EJv_1''v_2' + EJu_2''u_1' + {} \\ + EJv_2''v_1' - Lv_1'u_2' + Lu_1'v_2']_{z=l} = 0. \end{aligned} \qquad (2.62)$$

This condition is satisfied if we set

$$\left.\begin{aligned} EJu''(l) &= L[Au'(l) + kv'(l)], \\ -EJv''(l) &= L[(1 + k)u'(l) + Bv'(l)], \end{aligned}\right\} \qquad (2.63)$$

where A, B and k are constants. This can be verified by direct substitution.

Conditions (2.63) have a very simple mechanical meaning. Their left-hand sides contain the bending moments $L_y(l)$ and $L_x(l)$, equal to the projections onto the x and y axes of the moment L at the end section. Consequently Relations (2.63) specify a certain type of behavior of the moment L during bending deformation of the bar.

We shall now examine a few particular cases. Suppose that $A = B = k = 0$; then $L_x(l) = Lu'(l), L_y(l) = 0$. This corresponds to the case of a bar subjected to a couple $L = 2Pa$ applied to a rigid disk on the line $h = 0$ (Fig. 54a). It is assumed that in the state of torsion immediately preceding loss of stability the forces of the couple act tangentially to the disk, and that with loss of stability they retain their direction and point of application.

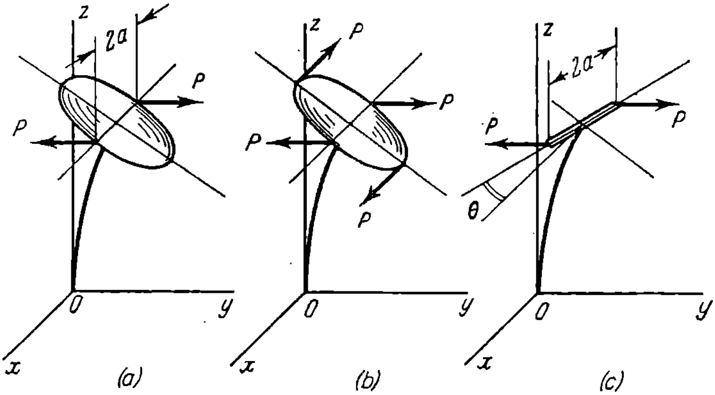


Fig. 54

Suppose now that $A = B = 0$ but that $k = -1/2$. In this case $L_x(l) = 1/2Lu'(l), L_y(l) = -1/2Lv'(l)$, i.e. the vector $L = 2Pa$ bisects the angle between the directions of the axes z and z_0. Such a moment can be produced by three or more equal and evenly spaced forces applied to the circumference of the disk and behaving in the same way as the forces in the previous case (Fig. 54b).

A further variant of the boundary conditions is obtained by putting $A = k = 0$, $B = \tan\theta$. Then $L_x = L(u' + \tan\theta v')$, $L_y = 0$. This is the case when two forces P are applied in the direction of the tangent to the disk in the undeformed state, their direction remaining constant during subsequent deformation. If θ is the angle of twist which occurs before loss of stability, then $L = 2Pa\cos\theta$ (Fig. 54c). If $\theta = 0$ we arrive back at the first of the three cases considered.

The foregoing types of twisting moment were classified by Ziegler, who called the first a *quasi-tangential* moment, the second a *semi-*

tangential moment and the third, a *pseudo-tangential* moment.[1] It is obvious that the moments in these problems are conservative, since the forces which form them are conservative.

Conservative stability problems for bars in torsion have been studied by E. L. Nikolai,[2] Shashkov,[3] Makushin,[4], Ziegler[5], Beck[6], and others. We shall not, however, go further into these problems, proceeding instead to the study of non-conservative problems.

A moment whose vector remains directed along the deformed axis of the bar can be produced by a pair of tangential forces (Fig. 55a).

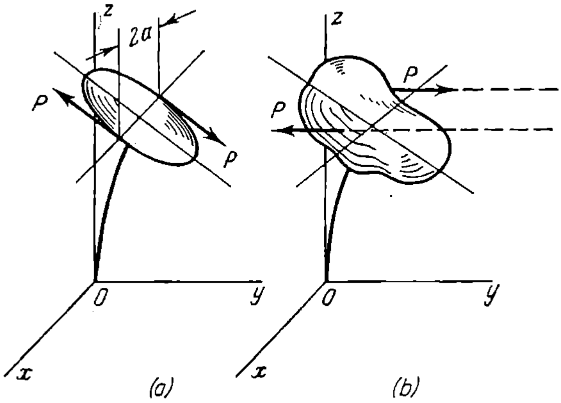


Fig. 55

We shall define this as a *tangential* moment. For this moment $L_x(l) = L_y(l) = 0$, and consequently

$$\Delta = -L\, v_1'(l)\, u_2'(l) + L\, u_1'(l)\, v_2'(l).$$

---

1. See reference on p. 89.

2. Nikolai, E. L. Studies in Mechanics (*Trudy po mekhanike*) Gostekhizdat. 1955.

3. Shashkov, I. E. On the stability of a bar of arbitrary section in compression and torsion. *Inzh. sborn.* 7 (1950).

4. Makushin, V. M. Investigation of the stability in torsion of a bar with equal principal bending stiffnesses. *Sb.* "*Raschety na prochnost'*", No. 2, Mashgiz, 1958; see also the book: Ponomarev, S. D., Biderman, V. L., Likharev, K. K., Makushin, V. M., Malinin, N. N., Feodos'ev, V. I. *Strength Analysis in Mechanical Engineering* (Raschety na prochnost' v mashinistroenii), Vol. III, Mashgiz, 1959.

5. Ziegler, H. Knickung gerader Stäbe unter Torsion. *Z. angew. Math. Phys.* 3, No. 2 (1952).

6. Beck, M. Knickung gerader Stäbe durch Druck und konservative Torsion, *Ing.-Arch.*, 23, No. 4 (1955).

If the end section can rotate, then with an arbitrary choice of functions u_1, u_2, v_1 and v_2 we have $\Delta \neq 0$. The boundary-value problem proves to be non-self-adjoint.

Consider a moment, the vector of which remains in a direction parallel to the undeformed axis. Such a moment could be produced by forces with fixed lines of action in space (an *axial* moment; see Fig. 55b). In this case $L_x(l) = L\,u'(l)$, $L_y(l) = -L\,v'(l)$ and consequently the quantity

$$\Delta = E J u_2'' u_1 + E J v_2'' v_1' = L\,u_1'(l)\,v_2'(l) - L\,v_1'(l)\,u_2'(l)$$

has the same value as for a tangential moment.

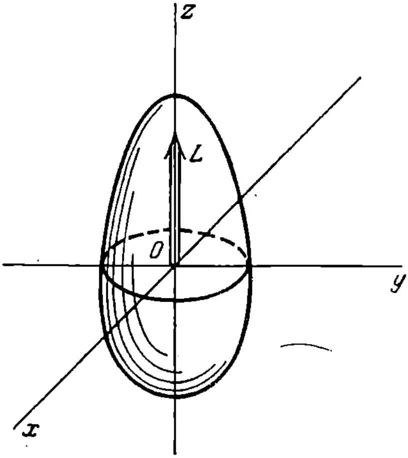


Fig. 56

The non-conservative nature of tangential and axial moments can also be seen from the following elementary considerations (Fig. 56). If we transpose a solid body, on which a tangential or axial moment L acts, into a new position, rotating it through an angle π about the z axis, the work done by the moment will be πL. We could have arrived at the same state by rotating the body first through an angle π about the axis Ox and then through an angle π about the axis Oy. The work done is zero both in the case of a tangential moment and in the case of an axial moment. We see then that the work done by the moment in passing from one state to the other depends on the path followed, which proves that the external forces have no potential.

We will show now that in the case of a tangential moment there can exist no deformed states of the bar in equilibrium close to the rectilinear form. Following the method proposed by E. L. Nikolai, let us consider the system of equations

$$EJu''' - Lv'' + Pu' = -\delta P_2,$$
$$EJv''' + Lu'' + Pv' = -\delta P_1.$$

We shall first find the general solution to these equations for $\delta P_1 \neq 0$, $\delta P_2 \neq 0$. Making use of the symmetry of the system, we introduce a complex deflection $w = u + iv$ and a complex force $\delta P = \delta P_1 + i\,\delta P_2$. In this way we obtain the equation

$$w''' - i\beta w'' + \alpha w' = -\delta p, \tag{2.64}$$

where

$$\alpha = \frac{Pl^2}{EJ}, \quad \beta = \frac{Ll}{EJ}, \quad \delta p = \frac{\delta P l^3}{EJ}, \quad \zeta = \frac{z}{l}. \tag{2.65}$$

The corresponding characteristic equation

$$r^3 - i\beta r^2 + \alpha r = 0,$$

apart from the root $r = 0$, has two purely imaginary roots: $r = i r_1$ and $r = i r_2$. Here

$$r_{1,2} = \frac{\beta}{2} \pm \sqrt{\left[\left(\frac{\beta}{2}\right)^2 + \alpha\right]}. \tag{2.66}$$

The general solution to Equation (2.64) is of the form

$$w = C_1 e^{ir_1} + C_2 e^{ir_2} + C_3 - \frac{\delta p \cdot \zeta}{\alpha}. \tag{2.67}$$

The boundary conditions for a bar clamped at one end and subjected at the other to a tangential moment and a compressive force are given by

$$w(0) = \frac{dw(0)}{d\zeta} = \frac{d^2 w(1)}{d\zeta^2} = \delta p = 0. \tag{2.68}$$

Hence we find

$$\left.\begin{aligned} &C_1 + C_2 + C_3 = 0: \qquad i r_1 C_1 + i r_2 C_2 = 0; \\ &\qquad r_1^2 C_1 e^{ir_1} + r_2^2 C_2 e^{ir_2} = 0. \end{aligned}\right\} \tag{2.69}$$

For this system of equations to have non-zero solutions, it is necessary and sufficient for its determinant to be zero, i.e.

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ i\,r_1 & i\,r_2 & 0 \\ r_1^2\,e^{i r_1} & r_2^2\,e^{i r_2} & 0 \end{vmatrix} = 0.$$

By definition r_1 and r_2 are real. Expanding the determinant, separating the real and imaginary parts, and taking the modulus of the complex expression, we find immediately that

$$|\Delta| = r_1\,r_2[r_1^2 + r_2^2 - 2r_1\,r_2\cos(r_1 - r_2)].$$

There are two possible ways of satisfying the condition $|\Delta| = 0$. The first is to take $r_1 = r_2$. Then, as follows from Formula (2.66),

$$\alpha = -\frac{\beta^2}{4}.$$

Thus the boundary-value problem has real values when P is a tensile force. Until now this result has passed unnoticed, although it can be obtained quite simply from relations derived by E. L. Nikolai.

The condition that $|\Delta| = 0$ will also be satisfied if $r_1 = -r_2$, $\cos 2r_1 = -1$. This corresponds to the assumption that $\beta = 0$ and $\alpha = n^2\pi^2/4$ $(n = 1, 2, 3, \ldots)$. Whence

$$P_* = \frac{\pi^2 E J}{4 l^2},$$

i.e. we arrive at a very familiar result. If $\beta \neq 0$, $\alpha > 0$, then the boundary-value problem has no real eigenvalues.

Analogous results can quite easily be obtained for an axial moment; in this case the third of boundary Conditions (2.68) should be replaced by $w''(1) = i\,\beta\,w'(1)$.

Since the present problem is non-conservative, the presence or absence of branch points in the static solutions should not be taken as an indication of stability or instability of the rectilinear form of equilibrium. For this reason we shall from now on investigate small oscillations of the bar about its position of equilibrium.

2.12. BAR WITH A CONCENTRATED MASS AT THE END. METHOD OF SMALL OSCILLATIONS

Suppose that a bar has a concentrated mass M at the end (Fig. 57). Ignoring the rotational inertia of this mass, we can write down approximate equations for small oscillations of the mass as follows:

$$\left.\begin{aligned} M\delta_{11}\frac{d^2u}{dt^2} + M\delta_{12}\frac{d^2v}{dt^2} + u = 0, \\ M\delta_{21}\frac{d^2u}{dt^2} + M\delta_{22}\frac{d^2v}{dt^2} + v = 0. \end{aligned}\right\} \qquad (2.70)$$

Here δ_{11} and δ_{12} are the displacements of the end of the bar in the directions of the axes Ox and Oy under the action of a unit concentrated force acting in the direction of Ox; δ_{21} and δ_{22} are the corresponding displacements due to a unit force acting in the direction of the axis Oy. These displacements are calculated, of course, taking into account the force P and the moment L.

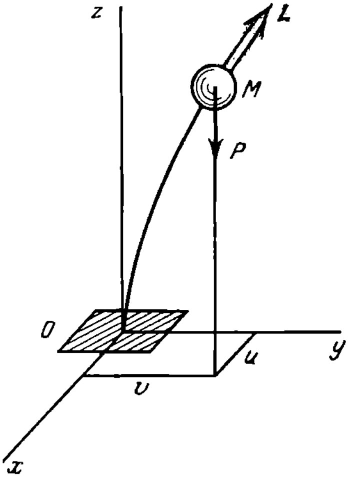


Fig. 57

In an effort to avoid tedious calculations we shall confine our attention to the case of a tangential moment, assuming that $P = 0$. Equation (2.64) then becomes

$$w''' - i\beta w'' = -\delta p,$$

where the definitions of (2.65) still apply. The characteristic equation

$$r^3 - i\beta r^2 = 0$$

has a purely imaginary root $r = i\beta$ and a double zero root. Thus the general solution to Equation (2.64) will be

$$w = C_1 e^{i\beta\zeta} + C_2\zeta + C_3 - \frac{\delta p \zeta^2}{2}.$$

If we now impose the boundary conditions $w(0) = w'(0) = w''(1) = 0$, we find

$$w(1) = \frac{\delta p}{\beta^2}\left[1 + \frac{\beta^2}{2} - (1 + i\beta)\, e^{-i\beta}\right]. \tag{2.71}$$

In order to find the unit deflections δ_{11} and δ_{12} we substitute the value of $\delta p = -l^3/EJ$ into Formula (2.71) and separate the real and imaginary parts. In this way we obtain

$$\delta_{11} = \operatorname{Re} w(1) = \frac{l^3}{\beta^2 EJ}\left(1 + \frac{\beta^2}{2} - \cos\beta - \beta\sin\beta\right),$$

$$\delta_{12} = \operatorname{Im} w(1) = -\frac{l^3}{\beta^2 EJ}(\sin\beta - \beta\cos\beta).$$

We determine the displacements δ_{21} and δ_{22} by setting $\delta p = -i\, l^3/EJ$. Thus

$$\delta_{21} = \operatorname{Re} w(1) = \frac{l^3}{\beta^2 EJ}(\sin\beta - \beta\cos\beta),$$

$$\delta_{22} = \operatorname{Im} w(1) = \frac{l^3}{\beta^2 EJ}\left(1 + \frac{\beta^2}{2} - \cos\beta - \beta\sin\beta\right).$$

Note that $\delta_{11} = \delta_{22}$ (which follows also from conditions of symmetry), but $\delta_{12} = -\delta_{21}$. This is of considerable significance for the problem of stability of the rectilinear form of equilibrium[1].

Substituting the expressions

$$u(t) = U e^{i\Omega t}, \qquad v(t) = V e^{i\Omega t},$$

into Equations (2.70), we obtain an equation for the natural frequencies

$$\begin{vmatrix} 1 - M\delta_{11}\Omega^2 & -M\delta_{12}\Omega^2 \\ -M\delta_{21}\Omega^2 & 1 - M\delta_{22}\Omega^2 \end{vmatrix} = 0.$$

If $\beta \neq 0$ the roots of this equation

$$\Omega = \pm\sqrt{\frac{\delta_{11} + \delta_{22} \pm \sqrt{[(\delta_{11} - \delta_{22})^2 + 4\delta_{12}\delta_{21}]}}{2M(\delta_{11}\delta_{22} - \delta_{12}\delta_{21})}},$$

---

[1] In order to reduce Equations (2.70) to a form analogous to the equations considered in Chapter 1, we simply introduce a matrix $b_{jk}(\beta)$ which is the inverse of the matrix $M\,\delta_{jk}(\beta)$. The equations then become

$$\frac{d^2 u_j}{dt^2} + \sum_{k=1,2} b_{jk}(\beta)\, u_j = 0.$$

Since $\delta_{12} = -\delta_{21}$, it follows that $b_{12} = -b_{21}$.

due to the properties of the unit displacements, always include two which have a negative imaginary part. Therefore the rectilinear form of the rod is unstable for any non-zero value of the moment L. This paradoxical result was first discovered by E. L. Nikolai. The explanation is, of course, that the system possesses multiple partial frequencies (when $\beta = 0$), and damping is not taken into account. If damping is taken into account the critical moment is found to be finite. It is also finite if the bar has unequal principal stiffnesses, in which case the partial frequencies are unequal, and the critical moment, therefore, is non-zero.

It is quite certain that non-conservative torsion occurs to some extent in all turbines. The fact that no actual cases of instability brought about by non-conservative torsion have been discovered must be explained, not so much by the effect of dissipative forces and slight ellipticity of the shaft, as by the effect of the conservative component of the moment. If a follower moment represents a sufficiently small addition to the conservative component, then it helps to stabilize the undisturbed form of equilibrium (or motion). This phenomenon, which was encountered in the problem of stability of compressed bars (in Section 2.5) and in the problem of stability of the plane form of bending (in Section 2.8), occurs also in the present problem.

2.13. EFFECT OF THE DISTRIBUTED MASS OF THE BAR AND OF DAMPING

By including inertia terms in Equations (2.57) we obtain the equations of small bending oscillations for a bar in compression and torsion:

$$\left.\begin{aligned} EJ\frac{\partial^4 u}{\partial z^4} + L\frac{\partial^3 v}{\partial z^3} + P\frac{\partial^2 u}{\partial z^2} + m\frac{\partial^2 u}{\partial t^2} &= 0, \\ EJ\frac{\partial^4 v}{\partial z^4} - L\frac{\partial^3 u}{\partial z^3} + P\frac{\partial^2 v}{\partial z^2} + m\frac{\partial^2 v}{\partial t^2} &= 0. \end{aligned}\right\} \qquad (2.72)$$

Making use of the complex deflection $w = u + i\,v$, we can write Equations (2.72) in the form

$$EJ\frac{\partial^4 w}{\partial z^4} - iL\frac{\partial^3 w}{\partial z^3} + P\frac{\partial^2 w}{\partial z^2} + m\frac{\partial^2 w}{\partial t^2} = 0. \qquad (2.73)$$

The boundary conditions for a bar, one end of which is fixed and the other subjected to a compressive force and an axial moment, are given by

$$w = \frac{\partial w}{\partial z} = 0 \quad \text{at} \quad z = 0,$$

$$E J \frac{\partial^2 w}{\partial z^2} - i L \frac{\partial w}{\partial z} = 0 \quad \text{at} \quad l = 0,$$

$$E J \frac{\partial^3 w}{\partial z^3} - i L \frac{\partial^2 w}{\partial z^2} + P \frac{\partial w}{\partial z} = 0 \quad \text{at} \quad l = 0.$$

The last condition can be interpreted as the requirement that the reactive forces δP_1 and δP_2 applied to the end of the bar are zero[1].

Making the substitution

$$w(z, t) = W(z) \sin(\Omega t + \varphi_0)$$

we reduce Equation (2.73) to the form

$$\frac{d^4 W}{d\zeta^4} - i\beta \frac{d^3 W}{d\zeta^3} + \alpha \frac{d^2 W}{d\zeta^2} - \omega^2 W = 0, \tag{2.74}$$

where the definitions given by (2.65) for the non-dimensional quantities still apply, as well as the additional definition

$$\omega^2 = \frac{m l^4}{E J} \Omega^2.$$

The boundary conditions are

$$\left.\begin{aligned} W(0) = \frac{dW(0)}{d\zeta} = \frac{d^2 W(1)}{d\zeta^2} - i\beta \frac{dW(1)}{d\zeta} = 0, \\ \frac{d^3 W(1)}{d\zeta^3} - i\beta \frac{d^2 W(1)}{d\zeta^2} + \alpha \frac{dW(1)}{d\zeta} = 0. \end{aligned}\right\} \tag{2.75}$$

and the characteristic equation is

$$r^4 - i\beta r^3 + \alpha r^2 - \omega^2 = 0.$$

Its roots will be denoted by r_1, r_2, r_3, r_4. Subjecting the solution to boundary Conditions (2.75), we obtain the following equation for the non-dimensional frequencies ω:

---

[1] See Equations (2.61).

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ (r_1^2 - i\beta r_1)\, e^{r_1} & (r_2^2 - i\beta r_2)\, e^{r_2} & (r_3^2 - i\beta r_3)\, e^{r_3} & (r_4^2 - i\beta r_4)\, e^{r_4} \\ \dfrac{1}{r_1} e^{r_1} & \dfrac{1}{r_2} e^{r_2} & \dfrac{1}{r_3} e^{r_3} & \dfrac{1}{r_4} e^{r_4} \end{vmatrix} = 0.$$

The difficulties involved in expanding this determinant and finding its roots are considerable; for this reason we shall confine our attention to the case of $\alpha = 0$. Since we expect that the critical values of the parameter β are fairly small, we shall assume that $\beta \ll \omega^2$. Then approximately, we have[1]

$$r = r_0 + \frac{i\beta}{4} + \cdots,$$

where r_0 is one of the roots of the equation $r^4 - \omega^2 = 0$. The general solution to Equation (2.74) is of the form

$$W(\zeta) = e^{\frac{i\beta\zeta}{4}} \sum_{j=1}^{4} C_j\, e^{r_0 \zeta} + \cdots.$$

Taking into account the boundary Conditions (2.75), we obtain an approximate equation (where $\lambda = \sqrt{\omega}$):

$$1 + \cos\lambda \cosh\lambda + \frac{i\beta}{2\sqrt{\omega}} (\cos\lambda \sinh\lambda + \sin\lambda \cosh\lambda) = 0. \quad (2.76)$$

Its smallest root is $\sqrt{\omega} = 1.875 + 0.144 i\beta + \cdots$. Thus for any non-zero value of β the frequency of oscillations Ω will contain an imaginary part, and the rectilinear form of equilibrium of the bar will be unstable.

An analogous result is obtained for the case of boundary Conditions II (Fig. 58). In the case of boundary Conditions III and IV loss of stability occurs by bifurcation of the forms of equilibrium (in spite of the fact that in case III the torsion is non-conservative).

We shall now consider briefly the effect of damping when $P = 0$. If in Equation (2.73) we add a term containing the first derivative with respect to time of the complex deflection, we obtain the equation

$$EJ \frac{\partial^4 w}{\partial z^4} - iL \frac{\partial^3 w}{\partial z^3} + m \frac{\partial^2 w}{\partial t^2} + m\varepsilon \frac{\partial w}{\partial t} = 0.$$

---

[1] We follow here the presentation of Troesch (see reference on p. 87).

We now substitute in this expression

$$w(z, t) = W(z)\, e^{st}.$$

The problem of investigating the stability of the undisturbed form of the bar reduces to a study of the complex eigenvalues ω of the boundary-value problem

$$\frac{d^4 W}{d\zeta^4} - i\beta \frac{d^3 W}{d\zeta^3} + \alpha \frac{d^2 W}{d\zeta^2} - \omega^2 W = 0 \qquad (2.77)$$

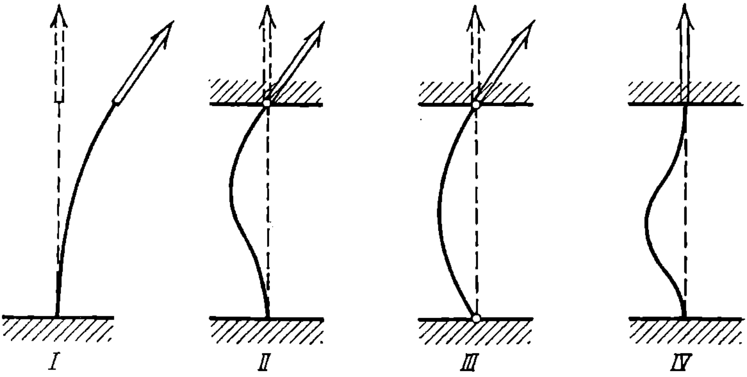


Fig. 58

with Conditions (2.75) at the ends. In this case

$$\omega^2 = -(\sigma^2 + g\,\sigma), \qquad (2.78)$$

where

$$\sigma = \frac{s}{\Omega_0}, \qquad g = \frac{\varepsilon}{\Omega_0}, \qquad \Omega_0 = \frac{1}{l^2}\sqrt{\frac{EJ}{m}}. \qquad (2.79)$$

The equilibrium is stable if all the characteristic exponents σ lie in the left-hand half-plane of the complex variable. The left-hand half-plane can be transformed by means of Formula (2.79) into the internal region of a parabola, the equation of which is given by

$$\operatorname{Im} \omega^2 = g(\operatorname{Re} \omega^2)^{1/2}. \qquad (2.80)$$

If $\operatorname{Im}\omega^2 < g(\operatorname{Re}\omega^2)^{1/2}$, the rectilinear form of the bar is stable. The area outside the parabola represents the region of instability of the rectilinear form of the bar (Fig. 59).

An exact analysis of the complex eigenvalues of the boundary-value problem requires very laborious calculations. We shall limit ourselves to an approximate solution, assuming that there is only slight damping and starting from a stability condition in the form $\operatorname{Re}\sigma < 0$, where

$$\sigma = -\frac{g}{2} \pm \sqrt{\left(\frac{g^2}{4} - \omega^2\right)}. \tag{2.81}$$

We see from Formula (2.81) that if there is only slight damping, or more exactly, if

$$g \ll (\operatorname{Re}\omega^2)^{1/2},$$

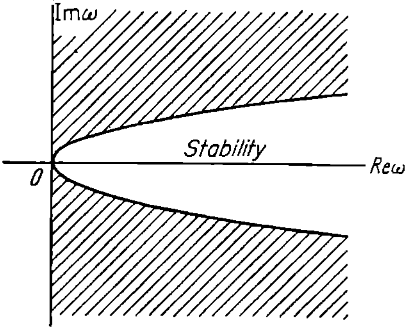


Fig. 59

the following inequality holds on the boundary of the stability parabola:

$$\operatorname{Im}\omega^2 \ll \operatorname{Re}\omega^2.$$

Thus from Formula (2.81)

$$\sigma \approx -\frac{g}{2} \pm i \operatorname{Re}\omega\left(1 + i\frac{\operatorname{Im}\omega}{\operatorname{Re}\omega}\right).$$

The stability condition now becomes

$$g > 2\operatorname{Im}\omega. \tag{2.82}$$

We have already given an Expression (2.76), for ω which is applicable for small moment parameters β. Using this expression, we find that $\operatorname{Im}\omega \approx 0.54\beta$, from which the critical value of the parameter $\beta_{**} \approx 0.93g$. As in the case of a system with two degrees of freedom (Section 1.16), we have established that for multiple partial frequencies the critical parameter is proportional to the degree of damping.

The foregoing calculations have been carried out for the case of an axial moment. Analogous results would be obtained if we considered the case of a tangential moment, replacing the third of the boundary Conditions (2.75) by the condition $W''(l) = 0$.

CHAPTER 3

STABILITY OF FLEXIBLE SHAFTS WITH CONTROLLED SPEED OF REVOLUTION

3.1. INTRODUCTORY REMARKS

Among the components of machines designed to transmit power the most important are shafts. As regards stability of shafts the following remarks are of some importance. Under constant conditions of loading the speed of rotation of a shaft can be kept constant within certain limits. In the most simple cases, when the changes in required power output are small and when they take place about some mean value, the revolutions can be controlled by a fly-wheel with a sufficiently large moment of inertia; in other cases a shaft can act as a link in an automatic control system. One of the problems facing the designer is to avoid dangerous transverse vibrations of the shaft induced by some part of the power in the rotational motion. A shaft with a controlled speed of revolution is a typical example of a *non-conservative system*, and the problem of the stability of the rectilinear (purely rotational) form of the shaft is a typical *non-conservative problem of elastic stability.*

The most simple example of this problem is illustrated by a weightless shaft of circular cross-section with one concentrated disk attached with its plane at right angles to the shaft (Fig. 60). The elementary theory for this case was given by Rankine, Laval and Stodola during the first stages of the development of steam turbines and reduces briefly to the following.

1. There exists a certain *critical speed* of rotation of the shaft at which the amplitudes of transverse vibrations resulting from eccentricity assume maximum values; this critical speed coincides with the frequency of natural oscillations of the non-rotating shaft

$$\Omega_0 = \sqrt{\frac{c}{M}} \tag{3.1}$$

(M is the mass of the disk, c the stiffness coefficient, equal to the force required to give unit deflection).

2. With further increase in angular velocity after transition through the critical value, the amplitude of the oscillations decreases, i.e. the disk tends towards a *self-centering state.* In actual rotors, which have not one but a whole spectrum of critical velocities, this self-centering process continues, of course, until the angular velocity approaches the next critical value. We see here the complete analogy with the problem of the oscillations of a usual elastic system under the action of periodic external forces.

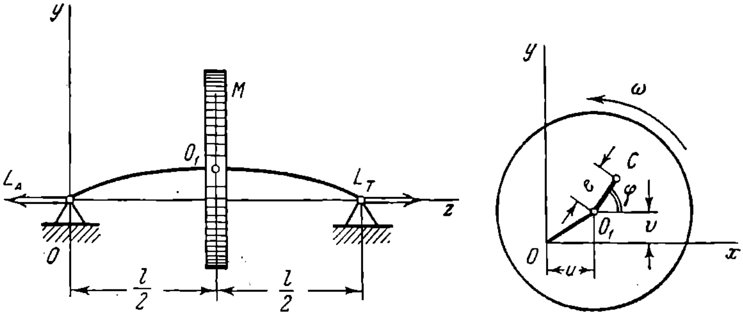


Fig. 60

Experiment shows, however, that flexible shafts often display a tendency to self-induced transverse vibrations at speeds other than the critical. These vibrations are certainly not associated with any eccentricity or other disturbing effects. The first explanation of self-induced vibrations which occur beyond the critical speed of rotation was given by Kimball[1], who was able to show the effect of internal friction.

Internal frictional forces within the shaft assist in damping vibrations if the shaft is rotating at a speed less than the lowest critical velocity. At velocities higher than the critical the internal frictional forces act in the direction of rotation and not in the opposite direction. If these forces are large enough compared with the forces of internal resistance, they can excite undamped vibrations.

---

[1] Kimball, A. L. Internal friction theory of shaft whirling. *Gen. Elec. Rev.*, 27, No. 4 (1924); Internal friction as a cause of shaft whirling. *Phil. Mag.*, ser. 6, 54 (1925).

The causes of self-excitation become clear if it is remembered that a shaft rotating at a constant angular velocity, independently of its state, is a non-conservative system. In the present problem there is an external energy source, a motor, which maintains a constant speed and transmits to the shaft the additional power necessary to maintain the vibrations. Internal friction acts as a transmitter of energy from the undisturbed (purely rotational) motion to the disturbed motion. We note the analogy here with the problem of stability of laminar flow of a viscous fluid, where the viscosity forces also contribute to the transmission of energy from the undisturbed (laminar) motion to the disturbed (turbulent) motion and are, therefore, an indirect cause of instability.

After the work carried out by Kimball, the theoretical aspects of the problem were investigated by Robertson[1], Smith[2], Nikolai[3], Kushul'[4], Leonov, Chaevskii and Bespal'ko[5] and others. The most thorough investigations are those carried out by Dimentberg[6]. A critical summary of the literature on this subject up until 1956 has been compiled by Pozniak[7].

[1] ROBERTSON, D. The vibrations of revolving shafts. *Phil. Mag.*, ser. 7, **13**, No. 8 (1932); Hysteretic influences on the whirling of rotors. *Proc. Inst. Mech. Engrs.*, **131** (1935).

[2] SMITH, D. M. Motion of a rotor carried by a flexible shaft in flexible bearings. *Proc. Roy. Soc.* **142**, No. 846A (1935).

[3] NIKOLAI, E. L. On the theory of flexible shaft. *Tr. Leningr. industr. in-ta, razd. fiz-mat. nauk*, No. 6, issue **3** (1937); see also NIKOLAI, E. L. *Studies in Mechanics* (Trudy no mekhanike), Gostekhizdat, 1955.

[4] KUSHUL', M. Ia. Transverse vibrations of rotating shafts in the presence of internal and external friction, *Izv. Akad. Nauk USSR*, No. 10 (1954).

[5] LEONOV, M. Ia. and CHAEVSKII, M. I. Experimental checking of the stability of rotation of shafts beyond their critical velocity. *Sb.* "*Voprosy mashinovedeniia i prochnosti v mashinostroenii*", No. 3, Izd-vo. Akad. Nauk USSR, 1955; LEONOV, M. Ia. and BESPAL'KO, L. A. On the investigation of the stability of a shaft rotating at a supercritical velocity, *ibid*; CHAEVSKII, M. I. Investigation of the stability of rotation of flexible shafts beyond the critical velocity. *Izv. Akad. Nauk USSR*, No. 9, 1955.

[6] DIMENTBERG, F. M. Transverse vibrations of a rotating shaft with disks attached, in the presence of frictional resistance. *Sb.* "*Poperechnye kolebani'a i kriticheskie skorosti*", No. 1. Izd-vo. Akad. Nauk USSR, 1951; On the stability of a flexible shaft with an unbalanced disk under the action of internal and external friction. *Izv. Akad. Nauk USSR*, OTN, No. 10 (1954); *Bending Oscillations in Rotating Shafts* (Izgibnye kolebaniia vrashcha'ushchikhsia valov). Izd-vo. Akad. Nauk USSR, 1959.

[7] POZNIAK, E. L. Influence of resistance on the stability of rotating shafts. *Sb.* "*Problemy prochnosti v mashinostroenii*", No. 1, Izd-vo. Akad. Nauk USSR, 1958.

The references listed above were concerned with the investigation of the stability of rotation with respect to small disturbances, and with finding the conditions under which self-excitation becomes possible. Recent research has been directed at finding the amplitudes of self-oscillations beyond the critical velocities; this requires the solution of non-linear equations[1].

Apart from internal frictional forces, there are a number of other factors such as aerodynamic, hydrodynamic and electromagnetic forces which can cause instability of a rotating shaft. Kapitsa[2] investigated instability caused by friction in bearings and by aerodynamic friction from the medium surrounding a rotor. Another familiar type of instability is that caused by hydrodynamic friction in the oil-film in bearings. This problem is of extreme importance in the design of modern high-speed turbines, and a great deal has been written on the subject.[3] Similar phenomena are encountered in rotors carrying a ferromagnetic core situated in a magnetic field and also in rotors containing a liquid with a free surface. Research into these problems has been carried out by Pozniak and Epishev.[4]

The present chapter deals in general terms with instability of rotors caused by the factors listed above; the close relation between these problems and other non-conservative problems of elastic stability will be illustrated. Considerable attention will be devoted to the non-linear treatment. Some use will also be made of the experimental results obtained by Pozniak and Epishev, published here for the first time.

---

[1] Bolotin, V. V. Investigation of self-oscillations of a flexible shaft caused by internal friction and allied effects. *Nauchn. dokl. vyssh. shkoly, ser. "Mashinostroenie i priborostroenie"*, No. 4 (1958); Non-linear vibrations of shafts beyond their critical speeds of rotation. *Sb. "Problemy prochnosty v mashinostroenii"*, No. 1, Izd-vo. Akad. Nauk USSR, 1958.

[2] Kapitsa, P. L. Stability and transition through the critical speed of revolving rotors in the presence of friction. *Zh. tekhn. fiz.*, **9**, No. 2 (1939).

[3] Newkirk, B. L. and Taylor, N. D. Shaft whirling due to oil action in journal bearings. *Gen. Elec. Rev.*, **28**, No. 8 (1925); Hagg, A. C. The influence of oil-film journal bearings on the stability of rotating machines. *J. Appl. Mech.*, **13**, No. 3 (1946); Poritsky, H. Contribution to the theory of oil whip, *Trans. Amer. Soc. Mech. Engrs.*, **75**, No. 6 (1953); Newkirk, B. L. and Lewis, J. F. Oil-film whirl—an investigation of disturbances due to oil-films in journal bearings. *Trans. Amer. Soc. Mech. Engrs.*, **78**, No. 1 (1956); Bocker, G. F. and Sternlicht, B. Investigation of translatory fluid whirl in vertical machines, *ibid.*

[4] Pozniak, E. L. Some questions on the stability of a steel core in a magnetic field. *Nauchn. dokl. vyssh. shkoly, ser. "Electromekhanika i avtomatika"*, No. 2 (1958); On the stability of an iron core in a magnetic field. *Izv. Akad. Nauk USSR*, No. 10 (1958); Stability of a rotating iron core in a magnetic field. *Izv. Akad. Nauk USSR, ser. Energetika i avtomatika*, No. 3 (1959); Epishev, L. V. On the dynamic instability of a revolving rotor partly filled with a liquid. *Nauchn. dokl. vyssh. shkoly, ser. "Mashinostroenie i priborostroenie"*, No. 2 (1959).

3.2. Equations of Motion of a Flexible Shaft

Let us derive the equation of motion for a flexible shaft carrying a concentrated disk (Fig. 60). We shall assume that the shaft is symmetrical, that the disk is attached at the center and that it vibrates only in a vertical plane, i.e. we shall not take into account the gyroscopic effect of the disk. In addition, we shall assume that the shaft is weightless and has an infinitely large torsional stiffness; this is equivalent to the supposition that the principal frequency of transverse natural oscillations of the shaft is much less than its higher frequencies and its frequency of natural torsional oscillations. The weight of the disk will not be taken into account, which corresponds either to the case of a vertical shaft or to the case of a sufficiently high speed of rotation. In these cases the effect of forces due to weight becomes negligibly small.

Let O_1 be the point at which the disk is attached to the shaft and C the center of mass of the disk. The distance $e = O_1 C$, called the eccentricity, is assumed to be small. Within the limits of the foregoing assumptions, the deformed state of the shaft will be fully defined if the coordinates u and v of the point of attachment of the disk (i.e. the components of the deflection of the shaft at midspan) and the angle φ which the line $O_1 C$ makes with the axis $O x$ are all known (Fig. 60).

To obtain the equations of motion of the disk we make use of d'Alembert's principle. The projections of the inertia forces on the axes $O x$ and $O y$ are given by

$$P_x = -M\left[\frac{d^2u}{dt^2} - \frac{d^2\varphi}{dt^2} e \sin\varphi - \left(\frac{d\varphi}{dt}\right)^2 e \cos\varphi\right],$$

$$P_y = -M\left[\frac{d^2v}{dt^2} + \frac{d^2\varphi}{dt^2} e \cos\varphi - \left(\frac{d\varphi}{dt}\right)^2 e \sin\varphi\right],$$

where M is the mass of the disk. The first terms in the brackets represents the linear acceleration, the second the angular acceleration and the third the centripetal acceleration.

The moment of the inertia forces about the axis $O z$ is given by the formula

$$L_z = -M\left(-\frac{d^2 u_c}{dt^2} v_c + \frac{d^2 v_c}{dt^2} u_c + r^2 \frac{d^2 \varphi}{dt^2}\right),$$

where r is the radius of gyration of the disk about an axis through the point C perpendicular to its plane, u_c and v_c are the coordinates of the point C, where

$$u_c = u + e \cos\varphi,$$
$$v_c = v + e \sin\varphi.$$

The equations of motion of the disk are

$$\left.\begin{aligned} M\frac{d^2 u}{dt^2} + c\,u &= M e\left[\frac{d^2\varphi}{dt^2}\sin\varphi + \left(\frac{d\varphi}{dt}\right)^2 \cos\varphi\right], \\ M\frac{d^2 v}{dt^2} + c\,v &= M e\left[-\frac{d^2\varphi}{dt^2}\cos\varphi + \left(\frac{d\varphi}{dt}\right)^2 \sin\varphi\right], \end{aligned}\right\} \tag{3.2}$$

$$M r^2 \frac{d^2\varphi}{dt^2} = M\left(\frac{d^2 u_c}{dt^2} v_c - \frac{d^2 v_c}{dt^2} u_c\right) + L_A - L_T. \tag{3.3}$$

Here L_A is the active (driving) moment transmitted to the shaft from the motor, L_T is the resistance moment and c is the stiffness coefficient of the shaft.

Consider the equation of rotary motion (3.3). We see from this that rotation at a constant angular velocity

$$\frac{d\varphi}{dt} = \omega = \text{const} \tag{3.4}$$

is possible if

$$L_A = L_T - M\left(\frac{d^2 u_c}{dt^2} v_c - \frac{d^2 v_c}{dt^2} u_c\right),$$

i.e. if the active moment varies in a suitable manner. Condition (3.4) is satisfied to some degree of accuracy in all automatically controlled machines. Even if the speed of rotation is controlled by a fly-wheel. Condition (3.4) is satisfied approximately, at any rate, for time intervals of quite considerable duration.[1]

---

[1] Systems with a limited power supply and falling characteristic are the subject of special consideration. See, for example: KONONENKO, V. O. On the transition through resonance of a mechanical system containing a motor. Sb. "*Problemy prochnosti v mashinostroenii*", No. 5, Izd-vo. Akad. Nauk USSR, 1959.

Suppose that the resistance moment exactly balances the active moment ($L_A = L_T$). Denoting the characteristic value of the amplitude of transverse vibrations by a, we note that $d^2u_c/dt^2 \sim a\,\omega^2$, $d^2v_c/dt^2 \sim a\,\omega^2$. Thus from Equation (3.4) we obtain

$$\frac{d^2\varphi}{dt^2} \sim \frac{e\,a}{r^2}\,\omega^2.$$

But $d^2\varphi/dt^2 = d\omega/dt$, and consequently, for the change in angular velocity $\Delta\,\omega$ during one revolution we find that

$$\frac{\Delta\,\omega}{\omega} \sim 2\pi\frac{e\,a}{r^2}. \tag{3.5}$$

According to the data presented by Biezeno and Grammel[1], eccentricities in steam turbines, even in the worst case, never exceed a few thousandths of the radius of gyration r of the disk. We shall therefore put $e = 10^{-3}r$. Suppose also that $a \sim 10e \sim 10^{-2}r$. We then find from (3.5) that

$$\frac{\Delta\,\omega}{\omega} \sim 10^{-4},$$

i.e. during the course of one revolution the angular velocity does not vary by more than 0.01 %.

By virtue of (3.4) Equations (3.2) can be written in the very much simplified form:

$$M\frac{d^2u}{dt^2} + c\,u = M\,e\,\omega^2\cos\omega\,t,$$

$$M\frac{d^2v}{dt^2} + c\,v = M\,e\,\omega^2\sin\omega\,t,$$

or, after substituting Ω_0, the frequency of natural oscillations, in the form

$$\left.\begin{aligned} \frac{d^2u}{dt^2} + \Omega_0^2\,u &= e\,\omega^2\cos\omega\,t,\\ \frac{d^2v}{dt^2} + \Omega_0^2\,v &= e\,\omega^2\sin\omega\,t. \end{aligned}\right\} \tag{3.6}$$

The self-centering effect of the disk after transition through the critical speed of rotation can be immediately established by considering the particular solution of Equations (3.6).

---

[1] Biezeno, C. B. and Grammel, R. *Engineering Dynamics*. Vol. 2, p. 225, Gostekhizdat, 1952. (English translation: Blackie, London 1955.)

3.3. VISCOUS INTERNAL FRICTION. INSTABILITY CAUSED BY INTERNAL FRICTION

As was pointed out by Kimball, in a consideration of the questions associated with instability in rotating shafts, the frictional forces must be divided into two categories. The first includes forces caused by contact of the rotating shaft with adjacent fixed components, as well as frictional forces from the surrounding medium if the latter can be considered to be stationary. The second category includes frictional forces set up by hysteresis in the material of the shaft, as well as certain other forces (for example, the frictional force between the shaft and a firmly seated disk). The distinguishing characteristic of the second category of forces is that they depend on the relative displacements and relative velocities of the rotating parts of the shaft, and consequently they must be *introduced by means of a rotating coordinate system.* Similar properties are to be found in the frictional forces in bearings containing an oil-film partially rotating with the shaft, and also in the forces between a rotor and medium partially rotating with the rotor. It is usual to define forces belonging to the first category as *external frictional forces*, and those of the second category as *internal frictional forces.*

Whereas external frictional forces always assist in damping the oscillations, internal frictional forces, under certain circumstances, can be the cause of instability, brought about in the end by transfer of part of the energy of rotation into energy of transverse vibrations. A very important feature of this phenomenon is to be found in the simple example of the case when the external and internal frictional forces are proportional to corresponding angular velocities of the disk.

Consider the case of the slightly disturbed motion of an ideally balanced shaft, when we must set $e = 0$ in Equations (3.6). In order to incorporate internal frictional forces proportional to velocities measured relative to the rotating shaft, we introduce a system of coordinates rotating at an angular velocity ω (Fig. 61). We pass from the new variables ξ, η to the old variables u, v by means of the formulae

$$\left.\begin{aligned} u &= \xi \cos\omega t - \eta \sin\omega t, \\ v &= \xi \sin\omega t + \eta \cos\omega t. \end{aligned}\right\} \qquad (3.7)$$

Applying the transformation (3.7) to Equations (3.6), we immediately obtain

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - 2\omega\frac{d\eta}{dt} + (\Omega_0^2 - \omega^2)\,\xi &= 0, \\ \frac{d^2\eta}{dt^2} + 2\omega\frac{d\xi}{dt} + (\Omega_0^2 - \omega^2)\,\eta &= 0. \end{aligned}\right\} \tag{3.8}$$

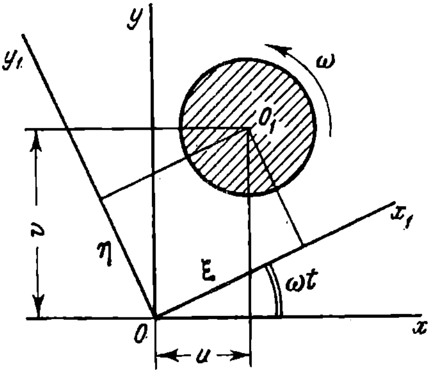


Fig. 61

We introduce into these equations the terms $\varepsilon_i(d\xi/dt)$ and $\varepsilon_i(d\eta/dt)$, which take into account internal frictional forces (ε_i is the coefficient of internal friction). Then, after returning to a fixed system of coordinates, we obtain the equations

$$\begin{aligned} \frac{d^2u}{dt^2} + \varepsilon_i\frac{du}{dt} + \Omega_0^2 u + \omega\,\varepsilon_i\, v &= 0, \\ \frac{d^2v}{dt^2} + \varepsilon_i\frac{dv}{dt} + \Omega_0^2 v - \omega\,\varepsilon_i\, u &= 0. \end{aligned}$$

We now have to introduce the external frictional forces $\varepsilon_e(du/dt)$ and $\varepsilon_e(dv/dt)$, which are proportional to the velocities of the disk in the fixed system of coordinates (ε_e is the coefficient of external friction). The equations of disturbed motion now become

$$\left.\begin{aligned} \frac{d^2u}{dt^2} + (\varepsilon_e + \varepsilon_i)\frac{du}{dt} + \Omega_0^2 u + \omega\,\varepsilon_i\, v &= 0, \\ \frac{d^2v}{dt^2} + (\varepsilon_e + \varepsilon_i)\frac{dv}{dt} + \Omega_0^2 v - \omega\,\varepsilon_i\, u &= 0. \end{aligned}\right\} \tag{3.9}$$

From the point of view of stability the presence in Equations (3.9) of pseudo-gyroscopic terms which comprise the antisymmetric matrix

$$\begin{Vmatrix} 0 & \omega\,\varepsilon_i \\ -\omega\,\varepsilon_i & 0 \end{Vmatrix}$$

is of considerable importance. In this sense Equations (3.9) are analogous to the equations of small oscillations of a bar subjected to non-conservative torsion, or alternatively, to the equations which describe the oscillations of a strip under the action of a "follower" force. The angular velocity of the shaft can be considered as a parameter to the accuracy of which the pseudo-gyroscopic terms are defined.

It follows that the problem of the stability of the rectilinear form of a rotating shaft under the effects of internal friction can be treated as a particular case of the non-conservative problems considered in Chapter 1. The present problem, however, has two distinguishing properties.

In the first place the pseudo-gyroscopic terms and the dissipative terms contain one and the same parameter. In this respect internal friction here induces two completely opposite tendencies—damping and instability, of which for sufficiently small angular velocities the former predominates, and for large angular velocities, the latter. It should be pointed out that certain other non-conservative problems of elastic stability possess the same property. For example, in the case of a body situated in a flow of gas, in addition to forces due to instability of the flutter type, the body is also subjected to forces of aerodynamic damping. At sufficiently high flow velocities destabilizing forces begin to dominate.

Secondly, in the case of a shaft of circular section there is a pair of multiple natural frequencies. Ω_0 and, in general, the whole spectrum consists of frequencies equal in pairs. Thus, what in the general theory was treated as an exception now becomes a rule. The problem remains non-conservative, of course, if the principal bending stiffnesses are not identical. This case will be dealt with briefly in Section 4.3.

Since in this problem both coordinates (x and y) are of equal importance, it is expedient to pass to a complex representation of the displacements. Introducing the complex displacement

$$w = u + i\,v,$$

we replace (3.9) by a single equation

$$\frac{d^2 w}{dt^2} + (\varepsilon_e + \varepsilon_i)\frac{dw}{dt} + \Omega_0^2 w - i\,\omega\,\varepsilon_i\,w = 0. \tag{3.10}$$

The solution to this equation is

$$w = a\,e^{i\Omega t}, \tag{3.11}$$

where Ω is the complex frequency.

The rectilinear form of the shaft is stable if $\operatorname{Im}\Omega > 0$. Substituting (3.11) into (3.10), and taking into account that on the boundary of the stability region $\operatorname{Im}\Omega = 0$, we obtain the two equations

$$\Omega_0^2 - \Omega^2 = 0, \qquad (\varepsilon_e + \varepsilon_i)\,\Omega - \omega\,\varepsilon_i = 0.$$

It follows from the first equation that $\Omega = \Omega_0$, i.e. that on the boundary of the region of instability oscillations take place at their natural frequency. From the second equation we obtain an expression for the critical velocity.[1]

$$\omega_* = \Omega_0\left(1 + \frac{\varepsilon_e}{\varepsilon_i}\right). \tag{3.12}$$

It should be noted that Formula (3.12) can be derived from the more general relations given in Chapter 1. With the definitions given in Section 1.16, taking into account that if $\beta = \omega/\Omega_0$, then

$$\omega_1 = \omega_2 = 1,$$

$$b_{12} = -b_{21} = \frac{\varepsilon_i}{\Omega_0^2},$$

$$\beta_0 = \frac{\Omega_0}{\varepsilon_i},$$

$$g_1 = g_2 = \frac{\varepsilon_e + \varepsilon_i}{\Omega_0},$$

and making use of Formula (1.96), we easily arrive at Formula (3.12). The graph of the variation of the characteristic exponents $s = i\,\Omega$, which is analogous to Figs. 5, 21, etc., is shown in Fig. 62.

---

[1] This formula can be found in the following references: SMITH, D. M. The motion of a rotor carried by a flexible shaft in flexible bearings. *Proc. Roy. Soc.*, Ser. A **142**, 92 (1933); DIMENTBERG, F. M. Transverse vibrations of a shaft with disks attached in the presence of frictional resistance. *Sb. "Poperechnye kolebaniia i kriticheskie skorosti"*, No. 1, Izd-vo. Akad. Nauk USSR, 1951.

According to Formula (3.12) the rectilinear form of an ideally balanced shaft becomes unstable at a velocity which exceeds the "classical" critical velocity Ω_0 by an amount which increases with increase in the ratio of external to internal friction. Qualitatively this formula accurately reflects the stabilizing role of the external frictional forces, but quantitatively it is not confirmed by experiment, due to the assumption made concerning the nature of the frictional forces.

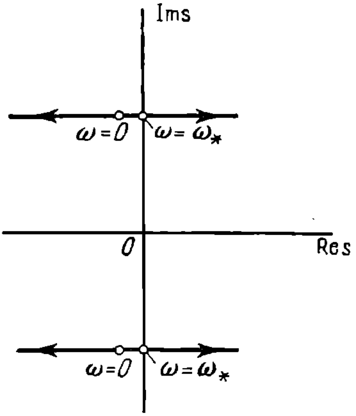


Fig. 62

3.4. Friction Independent of Velocity

It has been established by numerous experimental studies that the forces of internal friction within a material are practically independent of velocity, at any rate over a fairly wide range of frequencies. In other words, the area of the hysteresis loop is practically independent of the deformation rate. For this reason it is of particular interest to study the case when frictional forces are independent of velocity.

The detailed way in which the frictional forces vary during an oscillation period has very little effect on the process as a whole, but the resultant effect of frictional forces, measured by the energy dissipated over a cycle, is of paramount importance. In terms of hysteresis this means that damping depends principally on the area bounded by the hysteresis loop; the actual shape of the loop is of third-order importance. In studying the vibrations of shafts in the presence of frictional forces, it is therefore expedient to introduce the following integral characteristics—the magnitudes of the relative dissipation of energy due to internal and external friction during one cycle:

$$\psi_e = \frac{\Delta W_e}{W}, \qquad \psi_i = \frac{\Delta W_i}{W}. \tag{3.13}$$

Here ΔW_e and ΔW_i represent the absolute dissipation of energy during a cycle, and W is the total energy of the cycle, for example, the potential elastic strain energy at the maximum deflection from the equilibrium position. In the present section we shall consider the case when $\psi_e = \text{const}$ and $\psi_i = \text{const}$.

There are several ways in which the characteristics (3.13) can be introduced into the equations of oscillatory motion. One of them, indeed the simplest, is as follows.[1] Suppose first of all that the dissipative terms contain first derivatives of the displacements with respect to time multiplied by certain coefficients. For these terms to be independent of the frequency of vibrations, however, (or more precisely, for such dependence, if it exists, to appear exclusively in the expressions for ψ_e and ψ_i) these terms must be divided by the frequency. For example, in the case of forced oscillations of a system with one degree of freedom under the action of an harmonic force, the oscillations being expressed by the equation

$$M\frac{d^2u}{dt^2} + \varepsilon\frac{du}{dt} + c\,u = P\cos(\Omega\, t - \nu), \tag{3.14}$$

the coefficient ε must be taken in the form

$$\varepsilon = \frac{c\,\psi}{2\pi\,\Omega}, \tag{3.15}$$

where ψ is the relative dissipation of energy per cycle.

In fact Equation (3.14) has the solution

$$u = a\cos(\Omega\, t - \nu).$$

Thus the relative dissipation of energy over one cycle is

$$\Delta W = \int_0^{2\pi/\Omega} \frac{c\,\psi}{2\pi\,\Omega}\,\frac{du}{dt}\left(\frac{du}{dt}\,dt\right) = \frac{1}{2}\,\psi\, c\, a^2,$$

---

[1] PANOVKO, Ia. G. The effect of hysteresis losses in problems of the applied theory of elastic vibrations. *Zh. tekhn. fiz.* **23**, No. 3 (1953); SOROKIN, E. S. A method of taking into account non-elastic resistance of a material in vibration analyses of structures. *Sb. "Issledovaniia po dinamike sooryzhenii"*, Stroiizdat, 1951, and others.

whereas the total energy of the cycle is

$$W = \frac{1}{2} c\, a^2.$$

Consequently $\Delta W / W = \psi$, i.e. Equation (3.14) with the coefficient (3.15) does in fact describe oscillations which take place with a relative dissipation of energy ψ.

Note that when the coefficients of the inertia terms in the equations of motion are unity, we must replace (3.15) by the expression

$$\varepsilon = \frac{\Omega_0^2\, \psi}{2\pi\, \Omega}. \tag{3.16}$$

We shall now derive the equations of disturbed motion of an ideally balanced shaft when the internal and external frictional forces are independent of frequency of oscillation. We consider first the equations in a rotating system of coordinates:

$$\left.\begin{aligned} &\frac{d^2\xi}{dt^2} - 2\omega\frac{d\eta}{dt} + (\Omega_0^2 - \omega^2)\,\xi = 0, \\ &\frac{d^2\eta}{dt^2} + 2\omega\frac{d\xi}{dt} + (\Omega_0^2 - \omega^2)\,\eta = 0. \end{aligned}\right\} \tag{3.17}$$

We introduce into these equations terms which take into account internal friction. Let Ω be the complex frequency of oscillations in a fixed system of coordinates appearing in the solutions

$$u = u_0\, e^{i\Omega t}, \qquad v = v_0\, e^{i\Omega t}.$$

Since in the coordinate system rotating with the shaft the frequency is $\omega - \Omega$, then from Formula (3.16) the dissipative terms must be written in the form

$$\frac{\Omega_0^2\, \psi_i}{2\pi\, |\omega - \Omega|}\, \frac{d\xi}{dt}, \qquad \frac{\Omega_0^2\, \psi_i}{2\pi\, |\omega - \Phi|}\, \frac{d\eta}{dt}.$$

If we add these expressions, respectively, to the first and second of Equations (3.17), return to a fixed system of coordinates and then substitute the terms containing the external friction,

$$\frac{\Omega_0^2\, \psi_e}{2\pi\, \Omega}\, \frac{du}{dt}, \qquad \frac{\Omega_0^2\, \psi_e}{2\pi\, \Omega}\, \frac{dv}{dt},$$

we arrive at the equations

$$\left.\begin{aligned}\frac{d^2u}{dt^2}+\frac{\Omega_0^2}{2\pi}\left(\frac{\psi_e}{\Omega}+\frac{\psi_i}{|\omega-\Omega|}\right)\frac{du}{dt}+\frac{\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega-\Omega|}v+\Omega_0^2u=0,\\ \frac{d^2v}{dt^2}+\frac{\Omega_0^2}{2\pi}\left(\frac{\psi_e}{\Omega}+\frac{\psi_i}{|\omega-\Omega|}\right)\frac{dv}{dt}-\frac{\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega-\Omega|}u+\Omega_0^2v=0.\end{aligned}\right\}\qquad(3.18)$$

Here again the possibility of instability can be seen from the presence of pseudo-gyroscopic terms with the antisymmetric matrix

$$\begin{Vmatrix} 0 & \dfrac{\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega-\Omega|} \\ -\dfrac{\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega-\Omega|} & 0 \end{Vmatrix}.$$

Strictly speaking Formula (3.16) is valid only in the case of steady oscillations, when the frequency Ω is a real quantity. In the case when Ω is a complex quantity, but $\operatorname{Im}\Omega \ll \operatorname{Re}\Omega$, i.e. when the motion differs only slightly from a closed cycle, it can be looked upon as an approximate formula, and Equations (3.18), which are based upon it, can be treated as approximate equations. From now on, however, we shall use Equations (3.18) only to find the boundaries of regions of stability. On these boundaries we have the condition that $\operatorname{Im}\Omega = 0$, and consequently Equations (3.18) correspond completely to the assumptions on which Formula (3.16) is based.

Passing to complex variables, we obtain in place of (3.18) the single equation

$$\frac{d^2w}{dt^2}+\frac{\Omega_0^2}{2\pi}\left(\frac{\psi_e}{\Omega}+\frac{\psi_i}{|\omega-\Omega|}\right)\frac{dw}{dt}-\frac{i\,\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega-\Omega|}w+\Omega_0^2\,w=0. \qquad (3.19)$$

We shall seek a solution in the form

$$w = w_0\,e^{i\Omega t}. \qquad (3.20)$$

The rectilinear form of the shaft is stable if $\operatorname{Im}\Omega > 0$, and is unstable if $\operatorname{Im}\Omega < 0$. On the boundary of the region of stability $\operatorname{Im}\Omega = 0$, $\operatorname{Re}\Omega = \Omega$. Substituting (3.20) into (3.19) and equating in turn the real and imaginary parts of the result of this substitution to zero, we obtain the two equations

$$\Omega_0^2 - \Omega^2 = 0, \qquad \psi_e + \psi_i\frac{\Omega-\omega}{|\omega-\Omega|} = 0.$$

According to Formula (3.12) the rectilinear form of an ideally balanced shaft becomes unstable at a velocity which exceeds the "classical" critical velocity Ω_0 by an amount which increases with increase in the ratio of external to internal friction. Qualitatively this formula accurately reflects the stabilizing role of the external frictional forces, but quantitatively it is not confirmed by experiment, due to the assumption made concerning the nature of the frictional forces.

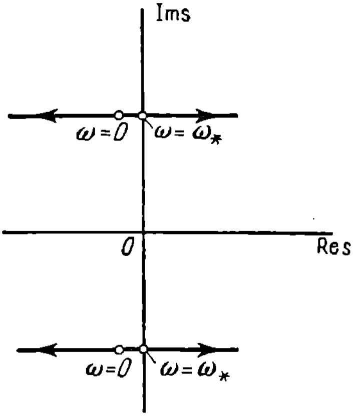


Fig. 62

3.4. Friction Independent of Velocity

It has been established by numerous experimental studies that the forces of internal friction within a material are practically independent of velocity, at any rate over a fairly wide range of frequencies. In other words, the area of the hysteresis loop is practically independent of the deformation rate. For this reason it is of particular interest to study the case when frictional forces are independent of velocity.

The detailed way in which the frictional forces vary during an oscillation period has very little effect on the process as a whole, but the resultant effect of frictional forces, measured by the energy dissipated over a cycle, is of paramount importance. In terms of hysteresis this means that damping depends principally on the area bounded by the hysteresis loop; the actual shape of the loop is of third-order importance. In studying the vibrations of shafts in the presence of frictional forces, it is therefore expedient to introduce the following integral characteristics—the magnitudes of the relative dissipation of energy due to internal and external friction during one cycle:

$$\psi_e = \frac{\Delta W_e}{W}, \qquad \psi_i = \frac{\Delta W_i}{W}. \tag{3.13}$$

Here ΔW_e and ΔW_i represent the absolute dissipation of energy during a cycle, and W is the total energy of the cycle, for example, the potential elastic strain energy at the maximum deflection from the equilibrium position. In the present section we shall consider the case when ψ_e = const and ψ_i = const.

There are several ways in which the characteristics (3.13) can be introduced into the equations of oscillatory motion. One of them, indeed the simplest, is as follows.[1] Suppose first of all that the dissipative terms contain first derivatives of the displacements with respect to time multiplied by certain coefficients. For these terms to be independent of the frequency of vibrations, however, (or more precisely, for such dependence, if it exists, to appear exclusively in the expressions for ψ_e and ψ_i) these terms must be divided by the frequency. For example, in the case of forced oscillations of a system with one degree of freedom under the action of an harmonic force, the oscillations being expressed by the equation

$$M \frac{d^2u}{dt^2} + \varepsilon \frac{du}{dt} + c\,u = P\cos(\Omega\, t - \nu), \tag{3.14}$$

the coefficient ε must be taken in the form

$$\varepsilon = \frac{c\,\psi}{2\pi\,\Omega}, \tag{3.15}$$

where ψ is the relative dissipation of energy per cycle.

In fact Equation (3.14) has the solution

$$u = a\cos(\Omega\, t - \nu).$$

Thus the relative dissipation of energy over one cycle is

$$\Delta W = \int\limits_0^{2\pi/\Omega} \frac{c\,\psi}{2\pi\,\Omega}\, \frac{du}{dt} \left(\frac{du}{dt}\,dt\right) = \frac{1}{2}\,\psi\, c\, a^2,$$

---

[1] PANOVKO, Ia. G. The effect of hysteresis losses in problems of the applied theory of elastic vibrations. *Zh. tekhn. fiz.* **23**, No. 3 (1953); SOROKIN, E. S. A method of taking into account non-elastic resistance of a material in vibration analyses of structures. *Sb. "Issledovaniia po dinamike sooryzhenii"*, Stroiizdat, 1951, and others.

whereas the total energy of the cycle is

$$W = \frac{1}{2} c\, a^2.$$

Consequently $\Delta W/W = \psi$, i.e. Equation (3.14) with the coefficient (3.15) does in fact describe oscillations which take place with a relative dissipation of energy ψ.

Note that when the coefficients of the inertia terms in the equations of motion are unity, we must replace (3.15) by the expression

$$\varepsilon = \frac{\Omega_0^2\, \psi}{2\pi\, \Omega}. \tag{3.16}$$

We shall now derive the equations of disturbed motion of an ideally balanced shaft when the internal and external frictional forces are independent of frequency of oscillation. We consider first the equations in a rotating system of coordinates:

$$\left.\begin{aligned} \frac{d^2\xi}{dt^2} - 2\omega \frac{d\eta}{dt} + (\Omega_0^2 - \omega^2)\, \xi &= 0, \\ \frac{d^2\eta}{dt^2} + 2\omega \frac{d\xi}{dt} + (\Omega_0^2 - \omega^2)\, \eta &= 0. \end{aligned}\right\} \tag{3.17}$$

We introduce into these equations terms which take into account internal friction. Let Ω be the complex frequency of oscillations in a fixed system of coordinates appearing in the solutions

$$u = u_0\, e^{i\Omega t}, \qquad v = v_0\, e^{i\Omega t}.$$

Since in the coordinate system rotating with the shaft the frequency is $\omega - \Omega$, then from Formula (3.16) the dissipative terms must be written in the form

$$\frac{\Omega_0^2\, \psi_i}{2\pi\, |\omega - \Omega|} \frac{d\xi}{dt}, \qquad \frac{\Omega_0^2\, \psi_i}{2\pi\, |\omega - \Phi|} \frac{d\eta}{dt}.$$

If we add these expressions, respectively, to the first and second of Equations (3.17), return to a fixed system of coordinates and then substitute the terms containing the external friction,

$$\frac{\Omega_0^2\, \psi_e}{2\pi\, \Omega} \frac{du}{dt}, \qquad \frac{\Omega_0^2\, \psi_e}{2\pi\, \Omega} \frac{dv}{dt},$$

we arrive at the equations

$$\left.\begin{aligned}\frac{d^2u}{dt^2}+\frac{\Omega_0^2}{2\pi}\left(\frac{\psi_e}{\Omega}+\frac{\psi_i}{|\omega-\Omega|}\right)\frac{du}{dt}+\frac{\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega-\Omega|}v+\Omega_0^2u=0,\\ \frac{d^2v}{dt^2}+\frac{\Omega_0^2}{2\pi}\left(\frac{\psi_e}{\Omega}+\frac{\psi_i}{|\omega-\Omega|}\right)\frac{dv}{dt}-\frac{\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega-\Omega|}u+\Omega_0^2v=0.\end{aligned}\right\}\qquad(3.18)$$

Here again the possibility of instability can be seen from the presence of pseudo-gyroscopic terms with the antisymmetric matrix

$$\begin{Vmatrix}0 & \dfrac{\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega-\Omega|}\\ -\dfrac{\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega-\Omega|} & 0\end{Vmatrix}.$$

Strictly speaking Formula (3.16) is valid only in the case of steady oscillations, when the frequency Ω is a real quantity. In the case when Ω is a complex quantity, but $\operatorname{Im}\Omega \ll \operatorname{Re}\Omega$, i.e. when the motion differs only slightly from a closed cycle, it can be looked upon as an approximate formula, and Equations (3.18), which are based upon it, can be treated as approximate equations. From now on, however, we shall use Equations (3.18) only to find the boundaries of regions of stability. On these boundaries we have the condition that $\operatorname{Im}\Omega = 0$, and consequently Equations (3.18) correspond completely to the assumptions on which Formula (3.16) is based.

Passing to complex variables, we obtain in place of (3.18) the single equation

$$\frac{d^2w}{dt^2}+\frac{\Omega_0^2}{2\pi}\left(\frac{\psi_e}{\Omega}+\frac{\psi_i}{|\omega-\Omega|}\right)\frac{dw}{dt}-\frac{i\,\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega-\Omega|}w+\Omega_0^2w=0.\qquad(3.19)$$

We shall seek a solution in the form

$$w = w_0\,e^{i\Omega t}.\qquad(3.20)$$

The rectilinear form of the shaft is stable if $\operatorname{Im}\Omega > 0$, and is unstable if $\operatorname{Im}\Omega < 0$. On the boundary of the region of stability $\operatorname{Im}\Omega = 0$, $\operatorname{Re}\Omega = \Omega$. Substituting (3.20) into (3.19) and equating in turn the real and imaginary parts of the result of this substitution to zero, we obtain the two equations

$$\Omega_0^2-\Omega^2=0,\qquad \psi_e+\psi_i\frac{\Omega-\omega}{|\omega-\Omega|}=0.$$

From the first equation, as in the case of viscous friction, we find that the frequency of oscillations $\Omega = \Omega_0$. The second equation can be re-written in the form

$$\psi_e - \psi_i \operatorname{sgn}(\omega - \Omega) = 0, \tag{3.21}$$

where sgn represents a change-of-sign function

$$\operatorname{sgn} x = \begin{cases} 1, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x = 0, \\ -1, & \text{if} \quad x < 0. \end{cases} \tag{3.22}$$

Equation (3.21) can be satisfied only if two conditions are fulfilled simultaneously. The first is that the angular velocity must be higher than the "classical" critical velocity Ω_0. The second condition is that the magnitudes of the relative dissipation of energy due to external and internal friction must be equal.

Thus if $\omega < \Omega_0$, the rectilinear form of an ideally balanced shaft is always stable. If $\omega > \Omega_0$, however, it can easily be shown that two cases can arise. If $\psi_i < \psi_e$ the rectilinear form is stable, and if $\psi_i > \psi_e$ it is unstable.

Equation (3.21) can be looked upon as an energetic relation between the forces of internal and external friction, the function sgn x representing the different effects of internal friction at pre- and post-critical velocities. If $\omega < \Omega_0$, internal friction acts together with external friction, i.e. in a direction opposite to the direction of motion (Fig. 63a). In this case it has a stabilizing effect. If $\omega > \Omega_0$ the internal and external frictional forces act in opposite directions, and consequently self-excitation of the vibrations becomes possible (Fig. 63b). The intermediate case, when $\omega = \Omega_0$, corresponds to the case of

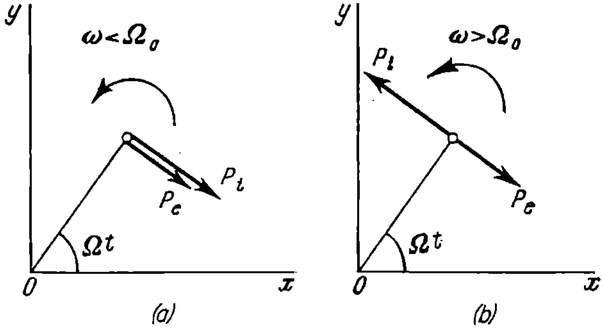


Fig. 63

so-called *synchronous* precession, when the deformed shaft rotates about a longitudinal axis without undergoing further deformations dependent on time, and therefore when internal friction is absent.

3.5. The Case of Arbitrary Dependence of Friction on Frequency[1]

The case when external and internal friction depends in an arbitrary manner on the frequency of the vibrations is of particular interest since, as can easily be shown, even when these forces are only slightly affected by the frequency, the nature of the problem in a qualitative sense can be altered quite considerably.

If the magnitudes of the relative dissipation of energy depend on the frequency, then Equations (3.18) remain valid; however, in these equations we must put

$$\psi_e = \psi_e(\Omega),$$
$$\psi_i = \psi_i(|\omega - \Omega|).$$

Following the procedure of the preceding section, we obtain

$$\Omega_0^2 - \Omega^2 = 0,$$
$$\psi_e(\Omega) - \psi_i(\omega - \Omega)\operatorname{sgn}(\omega - \Omega) = 0,$$

from which it immediately follows that the onset of instability occurs when the conditions

$$\omega > \Omega_0,$$
$$\psi_e(\Omega_0) < \psi_i(\omega - \Omega_0).$$

are satisfied.

These conditions can conveniently be represented graphically if we introduce the function

$$\psi(\omega) = \psi_e(\Omega_0) - \psi_i \operatorname{sgn}(\omega - \Omega_0),$$

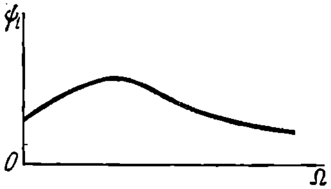


Fig. 64

---

[1] Pozniak, E. L. See reference on p. 141.

which, as we see, is the total relative dissipation of energy. Assume, for example, that ψ_e is independent of the frequency and that the relation $\psi_i = \psi_i(\Omega)$ is of the form shown in Fig. 64. The graph for the total dissipation of energy for this case is shown in Fig. 65. If $\omega < \Omega_0$ the external and internal losses add; if $\omega > \Omega_0$ they subtract. Here again a number of different cases can arise. If $\psi_e > \max\psi_i$, then the rectilinear form of the shaft remains stable for any post-critical velocity. If $\psi_e < \min\psi_i$, then everywhere beyond the "classical" critical velocity we shall have instability. If $\min\psi_i < \psi_e < \max\psi_i$, then at a certain angular velocity $\omega_* > \Omega_0$ instability occurs, but after the velocity ω_{**} is exceeded stability once more sets in (Fig. 65). This latter case is often encountered in everyday practice as well as in model tests.

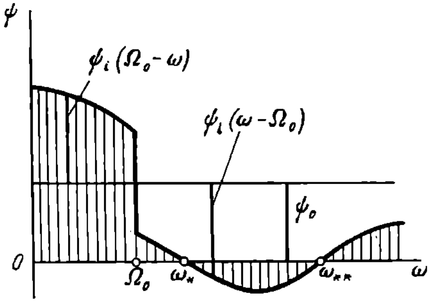


Fig. 65

The case considered in Section (3.3) of forces of viscous friction can be solved by using the general relations. If we express the relative dissipation of energy in terms of the coefficients ε_i and ε_e, we obtain

$$\psi_e = \frac{2\pi\,\varepsilon_e\,\Omega}{\Omega_0^2}, \qquad \psi_i = \frac{2\pi\,\varepsilon_i\,|\omega - \Omega|}{\Omega_0^2}.$$

A graph of the function $\psi = \psi(\omega)$ is shown in Fig. 66.

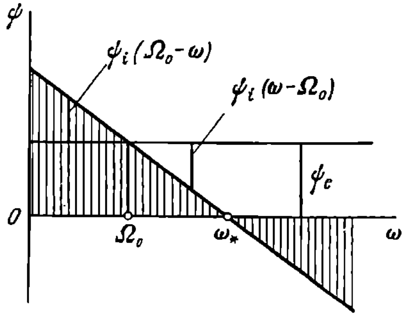


Fig. 66

3.6. GENERALIZATION OF THE PROBLEM TO THE CASE OF UNEQUAL PRINCIPAL STIFFNESSES AND AN INFINITE NUMBER OF DEGREES OF FREEDOM

In the preceding sections we have considered a shaft of circular cross-section. If the principal stiffnesses are not the same, the problem, of course, is still non-conservative. We shall make a brief study of this problem, confining our attention to the case of external and internal *viscous* friction.[1]

So as to obtain equations with constant coefficients we shall employ a coordinate system which rotates together with the shaft. Ignoring anisotropy of the coefficients of friction, in place of Equations (3.9) we obtain

$$\frac{d^2\xi}{dt^2} + (\varepsilon_e + \varepsilon_i)\frac{d\xi}{dt} - 2\omega\frac{d\eta}{dt} + (\Omega_1^2 - \omega^2)\,\xi - \varepsilon_e\,\omega\,\eta = 0,$$

$$\frac{d^2\eta}{dt^2} + (\varepsilon_e + \varepsilon_i)\frac{d\eta}{dt} + 2\omega\frac{d\xi}{dt} + (\Omega_2^2 - \omega^2)\,\eta + \varepsilon_e\,\omega\,\xi = 0.$$

Here Ω_1 and Ω_2 are the partial frequencies of the non-rotating shaft (evaluated, of course, without taking friction into account). The corresponding characteristic equation is given by

$$\begin{aligned} s^4 + 2(\varepsilon_e + \varepsilon_i)\,s^3 + [\Omega_1 + \Omega_2^2 + 2\omega^2 + (\varepsilon_e + \varepsilon_i)^2]\,s^2 + \\ + [(\varepsilon_e + \varepsilon_i)\,(\Omega_1^2 + \Omega_2^2 - 2\omega^2) + 4\varepsilon_e\,\omega^2]\,s + \\ + (\Omega_1^2 - \omega^2)\,(\Omega_2^2 - \omega^2) + \varepsilon_e^2\,\omega^2 = 0. \end{aligned}$$

Of the Routh-Hurwitz conditions, two are non-trivial for $\varepsilon_e > 0$, $\varepsilon_i > 0$. The first

$$(\Omega_1^2 - \omega^2)\,(\Omega_2^2 - \omega^2) + \varepsilon_e^2\,\omega^2 > 0$$

enables us to find the boundaries of the region of instability, which is situated in the interval between the partial frequencies. This form of instability is brought out by different principal stiffnesses; it has been studied in detail and we shall not consider it further here[2].

The second non-trivial condition

$$\begin{aligned} -2(\varepsilon_e + \varepsilon_i)^4\,(\Omega_1^2 + \Omega_2^2) + (\varepsilon_e + \varepsilon_i)^2\,(\Omega_2^2 - \Omega_1^2)^2 + \\ + 4\omega^2(\varepsilon_e + \varepsilon_i)^2\,[2(\Omega_1^2 + \Omega_2^2) + \varepsilon_i^2] - 16\varepsilon_i^2\,\omega^4 > 0 \end{aligned} \qquad (3.23)$$

---

[1] DIMENTBERG, F. M. See reference on p. 141.

[2] See, for example, the book: BOLOTIN, V. V. *Dynamic Stability of Elastic Systems* (Dinamicheskaia ustoichivost' uprugikh sistem). Gostekhizdat, 1956, which considers the corresponding non-linear problem.

corresponds to the specific instability caused by internal friction. If $\Omega_1 = \Omega_2 = \Omega_0$, then Condition (3.23) leads to a critical velocity given by Formula (3.12). If $\Omega_1 \neq \Omega_2$, but the internal and external frictions are sufficiently small, then we can use the simple formula

$$\omega_* \approx \frac{\Omega_1}{2}\left(1 + \frac{\varepsilon_e}{\varepsilon_i}\right)\sqrt{\left\{1 + \frac{\Omega_2^2}{\Omega_1^2} + \sqrt{\left[\left(1 + \frac{\Omega_2^2}{\Omega_1^2}\right)^2 + \frac{\left(1 - \frac{\Omega_2^2}{\Omega_1^2}\right)^2}{\left(1 + \frac{\varepsilon_e}{\varepsilon_i}\right)^2}\right]}\right\}}.$$

So far we have considered a weightless shaft with a single concentrated disk, treating it as a system with two degrees of freedom. To generalize this problem to the case of many or even an infinite number of degrees of freedom presents no serious difficulties, if the analytical expressions for terms which take into account internal and external friction are known.

As an example, consider the case of a shaft of constant section, assuming that the material of the shaft is visco-elastic. In this way the law governing forces of internal friction will be specified. We can therefore take the relation between the stress σ and the fiber strain ε as

$$\sigma = E\left(\varepsilon + \varkappa\frac{\partial \varepsilon}{\partial t}\right), \tag{3.24}$$

where $\varkappa$ is some constant which takes into account internal friction.

Consider first the case of a stationary shaft. Let $u(z, t)$, $v(z, t)$ be the projections of the displacements of an arbitrary point on the axis of the shaft onto the fixed axes of coordinates. Then, within the limitations of the assumption that plane sections remain plane, we obtain a formula for the longitudinal strain of an arbitrary fiber

$$\varepsilon = -\frac{\partial^2 u}{\partial z^2}y - \frac{\partial^2 v}{\partial z^2}x.$$

Taking into account (3.24), we readily obtain formulae for the bending moments on sections of the shaft

$$L_x = -EJ\left(\frac{\partial^2 v}{\partial z^2} + \varkappa\frac{\partial^3 v}{\partial z^2\,\partial t}\right),$$

$$L_y = EJ\left(\frac{\partial^2 u}{\partial z^2} + \varkappa\frac{\partial^3 u}{\partial z^2\,\partial t}\right).$$

The equations of natural oscillations for the non-rotating shaft become

$$E J \frac{\partial^4 u}{\partial z^4} + \varkappa E J \frac{\partial^5 u}{\partial z^4 \, \partial t} + m \frac{\partial^2 u}{\partial t^2} = 0,$$

$$E J \frac{\partial^4 v}{\partial z^4} + \varkappa E J \frac{\partial^5 v}{\partial z^4 \, dt} + m \frac{\partial^2 v}{\partial t^2} = 0,$$

where m is the mass per unit length of the shaft. We see then that inclusion of the effect of internal friction leads to the appearance of the additional terms

$$\varkappa E J \frac{\partial^5 u}{\partial z^4 \, \partial t}, \qquad \varkappa E J \frac{\partial^5 v}{\partial z^4 \, \partial t}.$$

in the oscillation equations.

Now consider the case of a rotating shaft. In this case the forces of internal friction must, obviously, be introduced in a rotating system of coordinates, and they must be expressed in terms of the projections of the displacements $\xi(z, t)$ and $\eta(z, t)$ on these coordinate axes (Fig. 66). We transform the equations

$$E J \frac{\partial^4 u}{\partial z^4} + m \frac{\partial^2 u}{\partial t^2} = 0,$$

$$E J \frac{\partial^4 v}{\partial z^4} + m \frac{\partial^2 v}{\partial t^2} = 0$$

to a rotating coordinate system. Setting

$$u = \xi \cos \omega t - \eta \sin \omega t,$$

$$v = \xi \sin \omega t + \eta \cos \omega t,$$

in these equations, we obtain

$$E J \frac{\partial^4 \xi}{\partial z^4} - 2\omega \frac{\partial \eta}{\partial t} + m \left(\frac{\partial^2 \xi}{\partial t^2} - \omega^2 \xi \right) = 0,$$

$$E J \frac{\partial^4 \eta}{\partial z^4} + 2\omega \frac{\partial \xi}{\partial t} + m \left(\frac{\partial^2 \eta}{\partial t^2} - \omega^2 \eta \right) = 0.$$

We include the additional terms

$$\varkappa E J \frac{\partial^5 \xi}{\partial z^4 \, \partial t}, \qquad \varkappa E J \frac{\partial^5 \eta}{\partial z^4 \, \partial t}$$

in these equations, return to a fixed coordinate system and then add the terms

$$\varepsilon_e \frac{\partial u}{\partial t}, \qquad \varepsilon_e \frac{\partial v}{\partial t},$$

which take into account viscous external friction. The final equations are found to be[1]

$$\left.\begin{aligned} EJ\frac{\partial^4 u}{\partial z^4} + \omega\varkappa EJ\frac{\partial^4 v}{\partial z^4} + \varkappa EJ\frac{\partial^5 u}{\partial z^4 \partial t} + \varepsilon_e \frac{\partial u}{\partial t} + m\frac{\partial^2 u}{\partial t^2} &= 0, \\ EJ\frac{\partial^4 v}{\partial z^4} - \omega\varkappa EJ\frac{\partial^4 u}{\partial z^4} + \varkappa EJ\frac{\partial^5 v}{\partial z^4 \partial t} + \varepsilon_e \frac{\partial v}{\partial t} + m\frac{\partial^2 v}{\partial t^2} &= 0. \end{aligned}\right\} \quad (3.25)$$

In the case of a shaft simply supported at the ends the system of Equations (3.25) has a particularly simple solution. Setting

$$u(z, t) = u_j(t) \sin\frac{j\pi z}{l},$$

$$v(z, t) = v_j(t) \sin\frac{j\pi z}{l}$$

$$(j = 1, 2, \ldots),$$

we see that the boundary conditions will be satisfied. Substitution into (3.25) yields

$$\frac{d^2 u_j}{dt^2} + (\varkappa\Omega_j^2 + \varepsilon_e)\frac{du_j}{dt} + \Omega_j^2 u_j + \omega\varkappa\Omega_j^2 v_j = 0,$$

$$\frac{d^2 v_j}{dt^2} + (\varkappa\Omega_j^2 + \varepsilon_e)\frac{dv_j}{dt} + \Omega_j^2 v_j - \omega\varkappa\Omega_j^2 u_j = 0$$

$$(j = 1, 2, \ldots),$$

where

$$\Omega_j = \frac{\pi^2}{l^2}\sqrt{\frac{EJ}{m}}.$$

If we omit the subscript j and put $\varkappa\Omega_j^2 = \varepsilon_i$, we obtain a system identical to (3.9). Hence, applying Formula (3.12) and returning to our previous notations, we find that the critical velocities are given by

$$\omega_*^{(j)} = \Omega_j\left(1 + \frac{\varepsilon_e}{\varkappa\Omega_j^2}\right) \qquad (j = 1, 2, \ldots).$$

---

[1] Equations of this type were first derived by DIMENTBERG (see, for example, the book: *Bending Oscillations of Rotating Shafts* (Izgibnye kolebaniia vrashchaiushchikhsia valov). Izd-vo. Akad. Nauk USSR, 1959.

It will be seen then that in the present problem there is no connection between the various modes of oscillations and instability, for a particular form occurs independently of the other modes of oscillations. In order for instability to occur the angular velocity must exceed the corresponding "classical" critical velocity (equal to the frequency of natural oscillations of the non-rotating shaft) by an amount which increases with increase in the external resistance.

The result is based on our assumptions concerning the nature of the internal and external frictional forces. It is not difficult to generalize this result to the case of an arbitrary relation between frictional forces and frequency, as was done in Section (3.5) for a shaft with one concentrated disk attached. Let Ω be the frequency of vibrations. Then we can take the following expression for the stresses

$$\sigma = E\left(\varepsilon + \frac{\psi_i}{2\pi\,|\omega - \Omega|}\,\frac{\partial \varepsilon}{\partial t}\right),$$

where ψ_i is an arbitrary function of the frequency in a rotating system of coordinates $\omega - \Omega$. With such a choice for the law governing internal friction, the relative dissipation of energy is the same for all forms of oscillations and is equal to ψ_i.

3.7. NON-LINEAR PROBLEM

So far we have investigated the stability of the undisturbed (i.e. purely rotational) motion with respect to small disturbances. It is of interest to investigate the non-linear problem for two reasons. In the first place, such an investigation enables us to find the amplitudes of self-oscillations which occur as a result of internal friction, and in this way to decide how dangerous these oscillations are and what measures can be taken to reduce them to acceptable values. Secondly, a non-linear investigation enables us to solve the problem of stability with respect to finite disturbances, i.e. the problem of stability "*in the large*". The fact that the question of stability "in the large" is of more than purely theoretical interest will be seen from what follows. As far back as 1924[1], it was noticed by Newkirk that a shaft, which maintains its rectilinear form beyond the critical velocity, starts to experience considerable vibrations if it is given a slight impact. Thus, motion which is stable "in the small" can be unstable "in the large". Transition to instability requires a disturbance which, although finite, is small enough from the practical point of view. We have here a practical case when it is insufficient to assess stability solely with respect to small disturbances.

---

[1] NEWKIRK, B. L. Shaft whipping. *Gen. Elec. Rev.*, 27, No. 3 (1924).

In the next few sections we shall make a non-linear study of the problem of vibrations in a flexible shaft of circular cross-section rotating at velocities beyond the critical[1]. Since our aim is to establish general principles associated with the effect of internal friction at post-critical velocities, we shall strive to simplify as far as possible the remainder of the problem. As in our analysis of the linear problem, we shall consider a weightless shaft with one attached disk, and we shall confine our attention to a mode of oscillation which does not allow the disk to warp and set up gyroscopic moments. This sort of shaft can be conveniently represented by our previous shaft simply supported at the ends with the disk at mid-span (see Fig. 60). We shall assume that the disk is ideally balanced initially, and in general we shall ignore the effect of its self-weight.

Let us consider the vibrations of the rotating shaft with finite amplitudes, treating the shaft initially as a system with a finite number

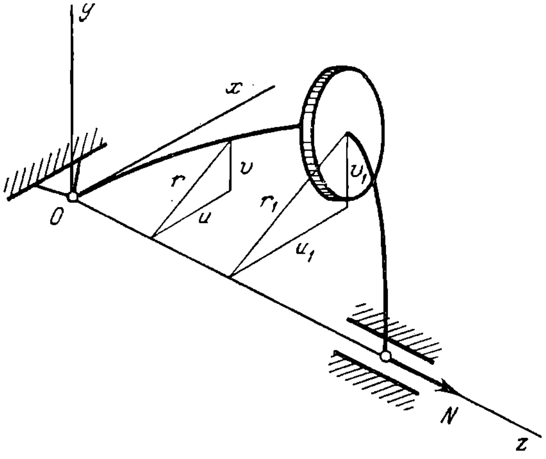


Fig. 67

---

[1] Bolotin, V. V. An investigation of self-oscillations of a flexible shaft induced by the effect of internal friction and related factors. *Nauchn. dokl. vyssh. shkoly, ser. "Mashinostroenie i priborostroenie"* No. 4 (1958); Non-linear oscillations of shafts beyond their critical speeds of rotation. *Sb. "Problemy prochnosti v mashinostroenii"*, No. 1, Izd-vo Akad. Nauk USSR, 1958.

of degrees of freedom (Fig. 67). In contrast to the elementary formulation of this problem, we shall employ the exact (non-linear) expressions for the curvature of the elastic line of the shaft, and we shall also take into account forces of inertia, elasticity and resistance resulting from axial displacements of points in the shaft. It can easily be shown that to take these forces into account is equivalent to taking into account non-linear terms in the corresponding components of the finite strain tensor (1.4).

Let u and v be the projections of the displacements of points on the shaft onto the fixed axes of coordinates Ox and Oy, $1/\varrho_x$ and $1/\varrho_y$ the projections of the curvature vector onto these axes, EJ the bending stiffness of the section of the shaft, m its mass per unit length and N a longitudinal force. Then the equations of oscillations of the shaft in a fixed system of coordinates become

$$\left.\begin{aligned}\frac{\partial^2}{\partial s^2}\left(\frac{EJ}{\varrho_x}\right)+\frac{\partial}{\partial s}\left(N\frac{\partial u}{\partial s}\right)+m\frac{\partial^2 u}{\partial t^2}=0,\\ \frac{\partial^2}{\partial s^2}\left(\frac{EJ}{\varrho_y}\right)+\frac{\partial}{\partial s}\left(N\frac{\partial v}{\partial s}\right)+m\frac{\partial^2 v}{\partial t^2}=0.\end{aligned}\right\}\tag{3.26}$$

Here s is the arc length measured along the deformed axis of the shaft (assumed inextensible); the terms which take into account external and internal friction will be introduced later. In the case when the deformed axis of the shaft is a plane curve we obtain

$$\frac{1}{\varrho}=\frac{\partial^2 r/\partial s^2}{\sqrt{[1-(\partial r/\partial s)^2]}},$$

where $r^2 = u^2 + v^2$. From this, taking into account that

$$\frac{1}{\varrho_x}=\frac{1}{\varrho}\,\frac{u}{r},\qquad \frac{1}{\varrho_y}=\frac{1}{\varrho}\,\frac{v}{r},\qquad \frac{\partial^2 r}{\partial s^2}=\frac{\partial^2 u}{\partial s^2}\,\frac{r}{u}=\frac{\partial^2 v}{\partial s^2}\,\frac{r}{v},$$

we find that

$$\frac{1}{\varrho_x}=\frac{\partial^2 u/\partial s^2}{\sqrt{[1-(\partial r/\partial s)^2]}},\qquad \frac{1}{\varrho_y}=\frac{\partial^2 v/\partial s^2}{\sqrt{[1-(\partial r/\partial s)^2]}}.$$

The longitudinal force N is composed of inertia forces due to the axial displacements of the disk $w(l/2)$, frictional forces in the fixed support and the reaction of the elastic coupling (Fig. 68). Let M be the mass of the disk, k the resistance coefficient and c the stiffness co-

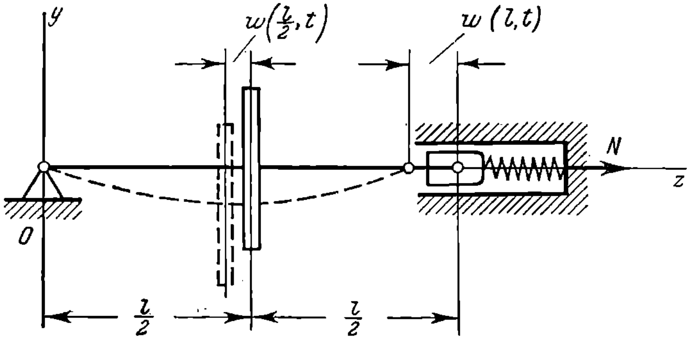


Fig. 68

efficient of the coupling. Then, assuming that the axis of the shaft is inextensible, we find

$$N(s,t) = \begin{cases} -M\dfrac{d^2w\left(\dfrac{l}{2},\ t\right)}{dt^2} - k\dfrac{dw(l,t)}{dt} - c\,w(l,t) \\ \qquad\qquad \left(0 \leqq s < \dfrac{l}{2}\right), \\ -k\dfrac{dw(l,t)}{dt} - c\,w(l,t) \qquad \left(\dfrac{l}{2} < s \leqq l\right). \end{cases}$$

Here $w(s, t)$ is the axial displacement of points on the shaft, i.e.

$$w(s,t) = s - \int_0^s \sqrt{\left[1 - \left(\frac{\partial r}{\partial s}\right)^2\right]}\,ds.$$

To the accuracy of second-order quantities

$$w(s,t) = \frac{1}{2}\int_0^s \left(\frac{\partial r}{\partial s}\right)^2 ds.$$

In order to pass from Equations (3.26) to ordinary differential equations, we set

$$u(s,t) = u_1(t)\sin\frac{\pi s}{l},$$

$$v(s,t) = v_1(t)\sin\frac{\pi s}{l},$$

$$r(s,t) = r_1(t)\sin\frac{\pi s}{l}$$

and apply Galerkin's variational method. In Equations (3.26), putting $m(s) = M\,\delta(l/2 - s)$, where δ is the delta function, we obtain

$$\left.\begin{aligned} \frac{d^2u}{dt^2} + \Omega_0^2 u + \Omega_0^2 u f\left(r, \frac{dr}{dt}, \frac{d^2r}{dt^2}\right) = 0, \\ \frac{d^2v}{dt^2} + \Omega_0^2 v + \Omega_0^2 v f\left(r, \frac{dr}{dt}, \frac{d^2r}{dt^2}\right) = 0. \end{aligned}\right\} \tag{3.27}$$

Here Ω_0 is the approximate value (in the Galerkin sense) of the frequency of natural small oscillations given by

$$\Omega_0^2 = \frac{\pi^4 E J}{2 M l^3},$$

f is a non-linear function, which to the accuracy of second-order quantities is of the form[1]

$$f\left(r, \frac{dr}{dt}, \frac{d^2r}{dt^2}\right) = \left(\frac{\pi^2}{8 l^2} + \frac{c\, l}{4\, E J}\right) r^2 + \\ + \frac{M l}{8 E J}\left[\left(\frac{dr}{dt}\right)^2 + r\frac{d^2r}{dt^2}\right] + \frac{k\, l}{2 E J} r \frac{dr}{dt}. \tag{3.28}$$

The indices of u_1, v_1 and r_1 have been omitted.

The first term on the right-hand side of (3.28) takes account of the non-linear expression for the curvature and the effect of the longitudinal elastic coupling, the second takes account of the inertia force of the disk due to its axial movement, the third term takes account of the frictional force in the longitudinal coupling. From now on we shall speak of non-linear elasticity, non-linear inertia and non-linear damping, respectively, writing the non-linear function in the form

$$f\left(r, \frac{dr}{dt}, \frac{d^2r}{dt^2}\right) = \gamma r^2 + 2\varkappa\left[\left(\frac{dr}{dt}\right)^2 + r\frac{d^2r}{dt^2}\right] + 2\varepsilon_0 r \frac{dr}{dt}. \tag{3.29}$$

Here

$$\gamma = \frac{\pi^2}{8 l^2} + \frac{c\, l}{4 E J}, \qquad \varkappa = \frac{M\, l}{4 E J}, \qquad \varepsilon_0 = \frac{k\, l}{4 E J}. \tag{3.30}$$

---

[1] A non-linear function of the type (3.28) in connection with the problem of shaft vibration for the case of unequal principal bending stiffnesses was given in: Bolotin, V. V. On the bending oscillations of flexible shafts with cross-sections of unequal stiffness *Inzh. sborn.*, **19**, izd-vo. Akad. Nauk USSR, 1954; *Dynamic Stability of Elastic Systems* (Dinamicheskaia ustoichivost' uprugikh sistem). Gostekhizdat, 1956.

We now introduce into Equation (3.27) terms which take account of internal and external friction. Transforming Equations (3.27) to a coordinate system which rotates together with the shaft, and substituting the terms

$$\frac{\Omega_0^2 \psi_i}{2\pi\,|\omega - \Omega|}\,\frac{d\xi}{dt}, \qquad \frac{\Omega_0^2 \psi_i}{2\pi\,|\omega - \Omega|}\,\frac{d\eta}{dt},$$

into the transformed equations, then returning to a fixed system of coordinates and substituting the terms

$$\frac{\Omega_0^2 \psi_e}{2\pi\Omega}\,\frac{du}{dt}, \qquad \frac{\Omega_0^2 \psi_e}{2\pi\Omega}\,\frac{dv}{dt},$$

we obtain

$$\left.\begin{aligned} &\frac{d^2u}{dt^2} + \frac{\Omega_0^2}{2\pi}\left(\frac{\psi_e}{\Omega} + \frac{\psi_i}{|\omega - \Omega|}\right)\frac{du}{dt} + \frac{\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega - \Omega|}\,v + \\ &\qquad + \Omega_0^2 u + \Omega_0^2 u f\left(r, \frac{dr}{dt}, \frac{d^2r}{dt^2}\right) = 0, \\ &\frac{d^2v}{dt^2} + \frac{\Omega_0^2}{2\pi}\left(\frac{\psi_e}{\Omega} + \frac{\psi_i}{|\omega - \Omega|}\right)\frac{dv}{dt} - \frac{\omega\,\Omega_0^2\,\psi_i}{2\pi\,|\omega - \Omega|}\,u + \\ &\qquad + \Omega_0^2 v + \Omega_0^2 v f\left(r, \frac{dr}{dt}, \frac{d^2r}{dt^2}\right) = 0. \end{aligned}\right\} \quad (3.31)$$

Here ψ_i and ψ_e are arbitrary functions of the amplitude and frequency, i.e.

$$\psi_i = \psi_i(r, |\omega - \Omega|), \qquad \psi_e = \psi_e(r, \Omega).$$

3.8. STEADY ASYNCHRONOUS PRECESSION

Among the solutions to the system of Equations (3.31), let us examine a particular class which in the case of a sufficiently well-balanced shaft is of the greatest interest from a practical point of view. These will be the solutions which correspond to steady asynchronous precession, i.e. to circular motion of the center of the disk at a constant amplitude and angular velocity which, in general, is not the same as the speed of rotation of the shaft (if it is the same the precession is synchronous). It was evidently this mode of self-oscillation that Newkirk observed in his widely known experiments on the stability of flexible shafts.

Let r be the amplitude and Ω the frequency of the oscillations (Fig. 69). Then the solutions mentioned above can be written in the form

$$u = r\cos\Omega t \qquad v = r\sin\Omega\, t. \tag{3.32}$$

Substituting (3.32) into the first of Equations (3.31), we obtain

$$(\Omega_0^2 - \Omega^2)r + \Omega_0^2\, r f(r, 0, 0) = 0, \tag{3.33}$$

$$\psi_e - \psi_i \operatorname{sgn}(\omega - \Omega) = 0. \tag{3.34}$$

Substitution into the second equation gives the same results; the reason for this becomes clear if we remember that Equations (3.31)

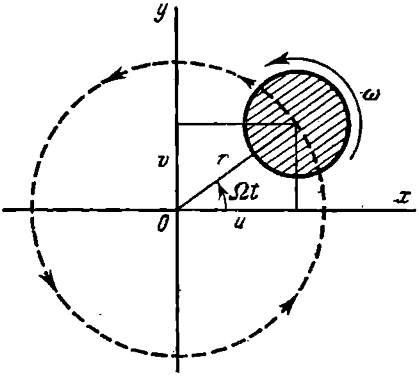


Fig. 69

are symmetrical with respect to u and v (or, more precisely, altering one to the other corresponds to reversing the direction of rotation) and that the required solution (3.32) is independent of the initial conditions.

We obtain the same results if we use complex variables for the displacements $w = u + i\,v$ and write the required solution in the form

$$w = r\,e^{i\Omega t}.$$

Substituting this expression into the equation

$$\frac{d^2w}{dt^2} + \frac{\Omega_0^2}{2\pi}\left(\frac{\psi_e}{\Omega} + \frac{\psi_i}{|\omega - \Omega|}\right)\frac{dw}{dt} - \frac{i\,\omega\,\Omega_0^2}{2\pi}\,\frac{\psi_i}{|\omega - \Phi|}\,w + {}$$
$$+ \Omega_0^2\, w + \Omega_0^2\, r f\left(r, \frac{dr}{dt}, \frac{d^2r}{dt}\right) = 0,$$

we readily obtain Relations (3.32) and (3.33).

It should be emphasized that Relations (3.32) and (3.33) give an exact solution to the non-linear system (3.31). This is made possible by the fact that the non-linearity is introduced into Equations (3.31) in the total deflection

$$r = \sqrt{(u^2 + v^2)},$$

which in the case of steady precession is independent of time.

Let us examine Equations (3.33) and (3.34) in more detail. The first is a relation between inertia and elastic forces; the second is a relation between internal and external friction. Suppose that $\psi_e > 0$, $\psi_i > 0$ and $f(r) > 0$ (this corresponds to the assumption that the dissipation of energy due to internal and external friction is positive, and that the non-linearity of the system is "hard"). With these rather unessential reservations, Equation (3.33) can be satisfied only if $\Omega > \Omega_0$ (if $r \neq 0$), and Equation (3.34) only if $\Omega < \omega$. It follows that the system has non-zero solutions only if $\omega > \Omega_0$.

Thus *self-excitation resulting from internal friction is possible only if* $\omega > \Omega_0$, *and the frequency of steady asynchronous precession lies within the limits* $\Omega_0 < \Omega < \omega$.

This conclusion is invalid if the non-linearity is "soft", since, as can be seen from (3.33), if $f(r) < 0$, solutions can exist also for $\omega < \Omega_0$. This, however, does not contradict the general conclusion that instability caused by internal friction is possible only in the post-critical region. The point is that in the case of "soft" non-linearity the natural frequency of oscillations with finite amplitudes is lower than the frequency of natural small oscillations Ω_0, which is usually looked upon as the critical velocity.

Equations (3.33) and (3.34) also enable us to find the upper limit of the frequency of vibrations during steady asynchronous precession. Indeed, since in this case we always have $\Omega < \omega$, it follows that $r < R$, where R is defined by the equation

$$\Omega_0^2 - \omega^2 + \Omega_0^2 f(R, 0, 0) = 0. \tag{3.35}$$

Thus, if $f(r) = \gamma r^2$, then

$$R = \frac{1}{\sqrt{\gamma}} \sqrt{\left(\frac{\omega^2}{\Omega_0^2} - 1\right)}. \tag{3.36}$$

Relation (3.36) is shown by the dotted line in Figs. 71, 73, 76, 82 etc.

Let us determine the order of possible amplitudes. Suppose initially that the movable support allows free axial displacement ($c = 0$). Then according to the first of Formulae (3.30)

$$\gamma = \frac{\pi^2}{8l^2}$$

and consequently

$$R = \frac{2\sqrt{(2l)}}{\pi}\sqrt{\left(\frac{\omega^2}{\Omega_0^2} - 1\right)}. \qquad (3.37)$$

Now consider the opposite case when both supports permit no displacement, i.e. when bending can occur only at the expense of axial extension of the bar. In this case we need to replace c in the formula by

$$c_0 = \frac{EF}{l},$$

where c_0 is the stiffness of the shaft in tension. From Formula (3.37) we then find

$$R = \frac{d}{2}\sqrt{\left(\frac{\omega^2}{\Omega_0^2} - 1\right)}$$

(here d is the diameter of the shaft).

We see then that the upper limit of the amplitude of vibrations of the shaft in a state of instability caused by internal friction varies form quantities of the order of the diameter (if the ends are fixed against axial displacement) to quantities of the order of the length of the shaft (if axial displacements are not inhibited). It is true that Formula (3.37) has been derived by retaining terms of an order not higher than the third, and in the case of large deflections gives too large results.

The results obtained correspond to only one of the possible regimes of self-oscillations. If the inequality $\psi_i > \psi_e$ holds for all values of Ω and r, Equation (3.34) cannot be satisfied, and therefore a regime of steady asynchronous precession is unattainable. It has been shown in a paper by the author[1] that in this case the oscillations will be of the beat type. The same paper investigates the stability of steady solutions.

---

[1] Bolotin, V. V. Non-linear oscillations of shafts beyond their critical speeds of rotation. *Sb. "Problemy prochnosti v mashinostroenii"* No. 1, Izd-vo. Akad. Nauk USSR, 1959.

An exact investigation of stability can to some extent be replaced by very simple "bifurcation" concepts. Already Poincaré[1], in his investigations of the forms of equilibrium of a rotating liquid established a very elegant theory for assessing stability purely from the shape of a "displacement-parameter" curve (in the present problem the displacement could be taken as the deflection r and the parameter as the angular velocity ω). Poincaré showed that if two branches of the curve intersect at some point, then *bifurcation (branching) of the forms of equilibrium* occurs at this point: a solution which is stable up to this point becomes unstable after it, and vice versa (see Fig. 70a where the stable branches are represented by thick lines and the unstable branches by thin lines). Another point at which transition from stability to instability also occurs is the *limiting point*, where the tangent to the curve becomes vertical (the points M_1 and M_2 in Fig. 70b). Bifurcation curves can also be used to investigate stability "in the large": the unstable branches can be looked upon as "waterdivide" which divide the direction followed by two very close stable solutions. This provides us with an easy method for finding the magnitude of the disturbance required to switch the system from one stable branch to another.

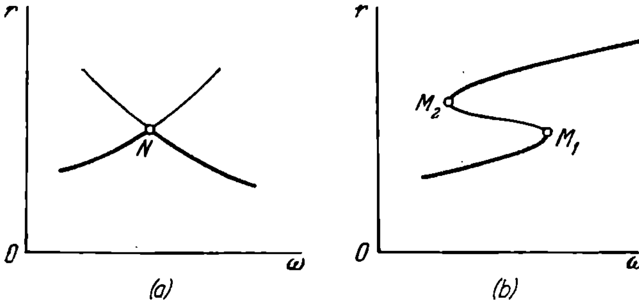


Fig. 70

Poincaré derived his theory in the first place to investigate the equilibrium of systems subjected to conservative forces. Andronov[2] developed it to cover the case of a self-oscillatory system with one degree of freedom, dependent upon a parameter and possessing slight non-linearity. He showed that all the principal theorems of Poincaré can be applied to this problem if the characteristic displacement is taken to mean the amplitude of the limit cycle. These results have recently been generalized by Neimark[1]. In the sequel, in order to select stable and unstable branches, we shall use bifurcation principles which have the advantage of being intuitively convincing.

---

[1] Appell, P. E. *Equilibrium Diagrams for a Rotating Homogeneous Liquid* (Figury ravnovesiia vrashchaiushcheisia odnorodnoi zhidkosti), ONTI, 1936.

[2] Andronov, A. A. *Application of Poincaré's theory of "bifurcation points" and "stability change" (échange des stabilités) to simple self-oscillatory systems*. Collection of papers. Izd-vo. Akad. Nauk USSR, 1956.

3.9. EXAMPLES OF AMPLITUDE RELATIONS

Proceeding now to a detailed analysis of the relation between amplitude and angular velocity after loss of stability, we shall first consider the case of linear friction:

$$\psi_e = \frac{2\pi \Omega \varepsilon_e}{\Omega_0^2}, \qquad \psi_i = \frac{2\pi |\omega - \Omega| \varepsilon_i}{\Omega_0^2}.$$

Equation (3.32) becomes

$$\frac{2\pi \Omega \varepsilon_e}{\Omega_0^2} - \frac{2\pi(\omega - \Omega) \varepsilon_i}{\Omega_0^2} = 0,$$

whence the frequency of self-oscillations is found to be

$$\Omega = \frac{\omega}{1 + \frac{\varepsilon_e}{\varepsilon_i}}. \tag{3.38}$$

From Equation (3.33) we find that

$$r = \frac{1}{\gamma} \sqrt{\left(\frac{\Omega^2}{\Omega_0^2} - 1\right)}.$$

Substituting (3.38) into this equation and taking into account that from Formula (3.12)

$$\Omega_0 \left(1 + \frac{\varepsilon_e}{\varepsilon_i}\right) = \omega_*,$$

we obtain

$$r = \frac{1}{\sqrt{\gamma}} \sqrt{\left(\frac{\omega^2}{\omega_*^2} - 1\right)}. \tag{3.39}$$

---

[1] NEIMARK, Iu. I. The method of point transforms in the theory of non-linear oscillations. *Izv. vyssh. uchebn. zaved., ser. Radiofizika* 1, Nos. 1, 2, 5, 6 (1958).

Relation (3.39) is plotted in Fig. 71. The greater the internal friction, the further away the region of instability of the rectilinear

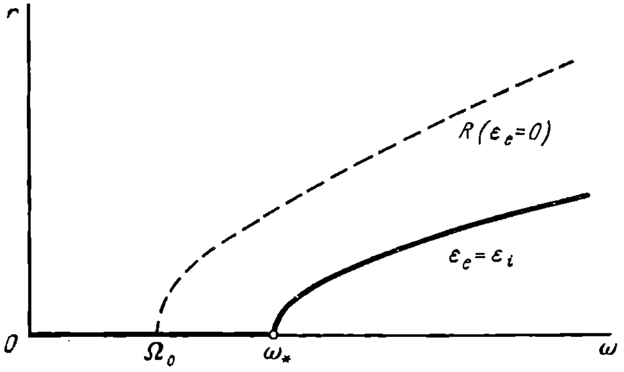


Fig. 71

form moves and the smaller the amplitudes of self-oscillations become. The dotted line represents the solution $r = R$ given by Formula (3.36).

So far we have assumed that the coefficients of friction ε_i and ε_e are independent of both the amplitude and the frequency of the oscillations. We shall now extend the theory to the case when

$$\varepsilon_i = k_i + K_i(r),$$
$$\varepsilon_e = k_e + K_e(r),$$

where k_i and k_e are constants and $K_i(0) = K_e(0) = 0$.

Formula (3.38) for this case assumes the form

$$\Omega = \frac{\omega}{1 + \dfrac{k_e + K_e(r)}{k_i + K_i(r)}}, \tag{3.40}$$

and we find from Equation (3.33) that

$$\Omega = \omega_0 \sqrt{[1 + f(r)]}. \tag{3.41}$$

Here the function $f(r)$ can be different in form from (3.29).

The amplitude curve can be constructed most easily by graphical methods. We simply find the points of intersection of the curves defined by Formulae (3.40) and (3.41). Fig. 72 shows the construction for the case when $f(r) = \gamma r^2$, $K_i = \alpha_i r^2$, $K_e = \alpha_e r^2$ ($\alpha_i > 0$, $\alpha_e > 0$). If the angular velocity is small enough there are no non-zero real roots. If $\omega = \omega_{**} > \Omega_0$, there exists a pair of multiple roots $r = r_{**}$. With further increase in the angular velocity, one of the roots increases; the other decreases until it vanishes when $\omega = \omega_*$. If $\omega > \omega_*$ there is only one real root.

The realtion between amplitude of the vibrations and angular velocity is shown in Fig. 73. N is the point of bifurcation (here $\omega = \omega_*$,

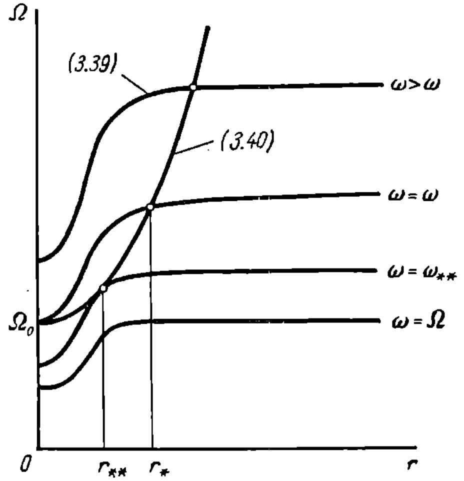


Fig. 72

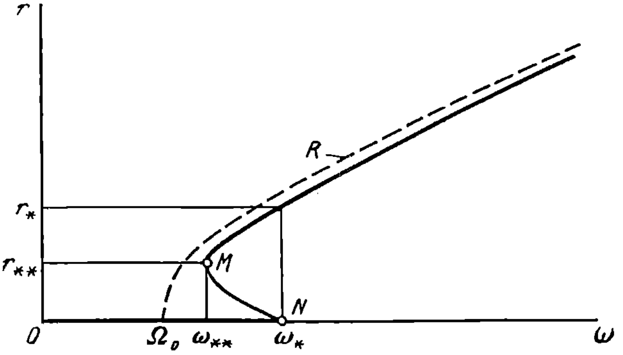


Fig. 73

where $\omega = \omega_*$ is the critical velocity according to the linear theory). M is the limiting point: on passing through this point the unstable solution NM becomes stable (here, and also in subsequent diagrams, stable solutions are denoted by thick lines).

If the angular velocity is gradually increased from zero, the following events take place. The rectilinear form of the shaft remains stable while $\omega < \omega_*$. When $\omega = \omega_*$, "hard" disturbance of the oscillations occurs with a steady finite amplitude r_*. With further increase in angular velocity the amplitude increases (Fig. 74a). If the angular velocity is decreased, the oscillation regime is maintained as far as $\omega = \omega_{**}$, where the oscillations "break off" and the initial undistorted form of the shaft is restored (Fig. 74b). Here we have a case of "entrainment", typical of many problems of non-linear oscillations.

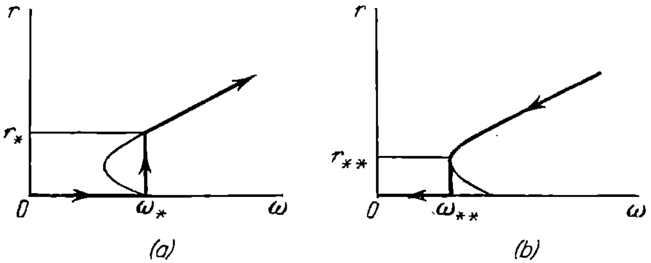


Fig. 74

The above remarks refer to the case when the disturbances acting on the shaft are sufficiently small. If, however, the disturbances are not small we must consider the likelihood of loss of stability "in the large". For example, if a rectilinear shaft working within the range $\omega_{**} < \omega < \omega_*$ is given a sufficiently strong lateral impact, the shaft might not return to its initial position, although it is in fact stable. On the other hand, a shaft vibrating in the interval $\omega_{**} < \omega < \omega_*$ can be returned to its rectilinear form by imparting a large enough disturbance. The order of the disturbance needed to cause transition from one stable branch to another depends on the position of the unstable branch which acts, as already mentioned, as a water-divide. Roughly speaking we have to "throw" the system through the unstable branch.[1]

---

[1] Here we are referring to disturbances of one specific type (amplitude disturbances). In order to answer this question completely we should have to consider the phase trajectories of the system in a four-dimensional space, which in practice would hardly be possible.

An analysis of the non-linear problem leads once more to an investigation of the problem of the stability of the rectilinear form of the shaft. It appears that a consideration of the "variational equations" for the zero solution is still insufficient to enable us to assess the likelihood of instability. A shaft which is stable with respect to extremely small disturbances can prove to be unstable with respect to finite disturbances, although they may be quite small from the practical point of view.

This will be made clear from the following example. Suppose that the internal friction has no linear part, i.e. $k_i = 0$ and $K_i(r) = \alpha_i r^2$. This case is very similar to what we have in practice (the area of the hysteresis loop tends to zero as the stresses become infinitely small). Then, calculating the critical velocity from Formula (3.12), we find that $\omega_* \to \infty$, i.e. that the shaft remains stable for any angular velocity.

This case is shown graphically in Fig. 75 and the results are shown in Fig. 76. The rectilinear form of the shaft, stable with respect to small disturbances for all angular velocities starting from $\omega = \omega_{**}$ can prove to be unstable "in the large". The magnitude of the disturbance required to cause vibration will be smaller the higher the angular velocity.

The velocity $\omega = \omega_{**}$ at which the rectilinear form of the shaft ceases to be the only stable form, can be considered as the "lower"

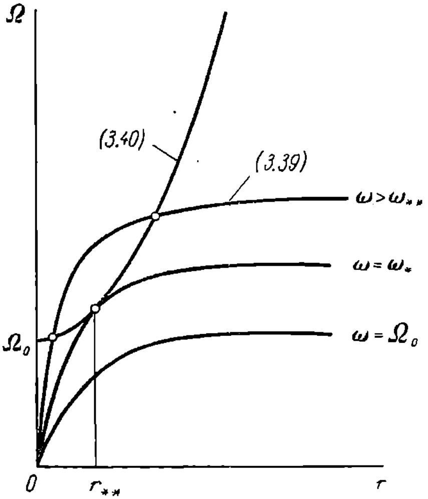


Fig. 75

critical velocity (in contrast to the "upper" critical velocity ω_*). We have here a very close analogy with the theory of flexible shells, where

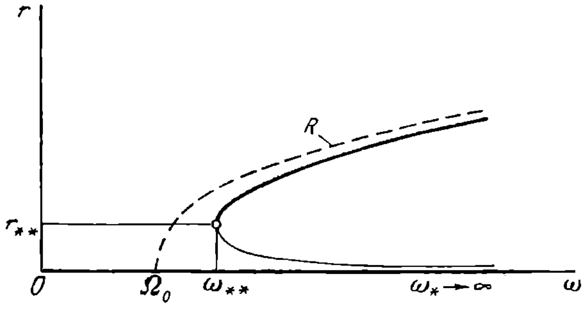


Fig. 76

the "lower" critical load assumes considerable practical importance and the "upper" critical load, determined from the linear theory, to a large extent loses its significance.

3.10. FRICTION CAUSED BY MACROSCOPIC THERMAL DIFFUSION

In what follows, instead of considering various laws governing the behavior of external and internal friction, we shall simply select those which correspond most closely to experimental observations.

A large part of all internal losses results from irreversible processes associated with macroscopic thermal diffusion. We shall look at this problem in a little more detail.

It is an established fact that when a bar made of a material with a positive thermal expansion coefficient is extended it cools slightly, and when it is compressed its temperature rises (this can be proved from purely thermodynamic concepts). Therefore, if a bar undergoes bending oscillations, fibers will be subjected to alternate rises and falls in temperature. There will consequently be a flow of heat from the heated (compressed) fibers to the cooled (extended) fibers which, as a result of the periodicity of the temperature changes, is also periodic. The process of heat transfer from a cold to a hot region is not reversible, and the oscillations are therefore accompanied by a certain dissipation of energy: part of the mechanical energy of the elastic oscillations is transformed into heat and dissipated.

Zener[1] considered the process of heat diffusion during bending oscillations and derived the formula

$$\psi_i = 2\pi \frac{E_S - E_T}{\sqrt{(E_S E_T)}} \frac{\omega_r \Omega}{\omega_r^2 + \Omega^2}. \tag{3.42}$$

Here E_S and E_T are the adiabatic and isothermal moduli of elasticity, ω_r is the relaxation frequency, which for a bar of circular section is given by

$$\omega_r \approx \frac{D}{d^2} \tag{3.43}$$

(D is the coefficient of macroscopic thermal diffusion).

Relation (3.42) is shown in Fig. 77. In the case of very small frequencies ($\Omega \ll \omega_r$) the process takes place almost adiabatically.

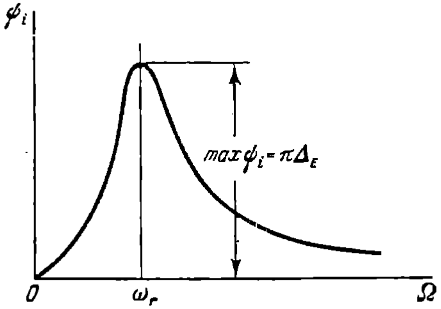


Fig. 77

Conversely if $\Omega \gg \omega_r$ the temperature barely has time to change, i.e. the process is almost isothermal. In both cases loss of energy due to thermal diffusion is extremely small, as can be seen from Fig. 77. The greatest losses take place when $\Omega = \omega_r$ and are given by

$$\max\psi_i = \pi \Delta_E, \tag{3.44}$$

where Δ_E is the so-called degree of relaxation

$$\Delta_E = \frac{E_S - E_T}{\sqrt{(E_S E_T)}} \approx \frac{E_S - E_T}{E_S}.$$

---

[1] Zener, C. *Elasticity and Anelasticity of Metals*, University of Chicago Press, Chicago 1948.

It follows from thermodynamic principles that

$$\Delta_E = E_S T \frac{\alpha^2}{c}, \qquad D = \frac{\lambda}{\varrho c},$$

where T is the absolute temperature, c the heat capacity, λ the coefficient of heat conduction, α the coefficient of linear expansion and ϱ the density.

For steel these formulae give (at room temperature)

$$\Delta_E = 0.0024, \qquad D = 0.20\,\text{cm}^2/\text{sec}.$$

The values found for ψ_i from these results coincide satisfactorily with the experimental results of Benewitz and Rötger[1]. Substituting these values into Formulae (3.43) and (3.44), we obtain (with $d = 20$ mm) $\max \psi_i \approx 0.0075$, $\omega_r = 0.05$l/sec.

The fact that the relaxation frequency is extremely small would seem to indicate that these effects cannot significantly influence the stability of the shaft. This, however, is not so. Since internal friction is associated with deformation of the shaft, i.e. depends on its behavior in a rotating system of coordinates, we must replace Ω in Formula (3.42) by the frequency $|\omega - \Omega|$. In other words, the relaxation frequency must be compared not with ω_r but with $|\omega - \Omega|$.

It follows from Equations (3.33) that $\Omega = \omega_0 \sqrt{[1 + f(r)]}$. But in the case of small oscillations $f(r) \ll 1$, $\Omega \approx \Omega_0 \approx \omega$. It will be seen immediately that at least on the boundary of the region of stability $|\omega - \Omega| \sim \omega_r$.

Macroscopic thermal diffusion only partially explains the dissipation of energy during vibration. If the stresses are large enough, losses of energy due to irreversible plastic strains and elastic hysteresis begin to play an important part. These losses, which are independent of the frequency of vibrations, depend on the peak values of the stresses, i.e. they depend ultimately on the vibration amplitude r. Therefore the total expression for the internal dissipation of energy can by written in the form

$$\psi_i = C r^n + 2\pi \Delta_E \frac{\omega_r |\omega - \Omega|}{(\omega - \Omega)^2 + \omega_r^2}. \tag{3.45}$$

Here C and n are constants found experimentally. According to a number of results the power index n for steel has a value somewhere between 2 and 3.

---

[1] See reference on p. 177.

The problem of finding the amplitudes and frequencies for steady asynchronous precession reduces to one of solving Equations (3.33) and (3.34) on the assumption (3.45). For brevity, setting

$$(\omega - \Omega)^2 + \omega_r^2 = F, \tag{3.46}$$

we can write Equation (3.34) in the form

$$C\,r^n + 2\pi\,\Delta_E\,\frac{\omega_r\sqrt{(F - \omega_r^2)}}{F} = \psi_e.$$

Solving this equation for F, we find

$$F = \frac{2\pi\,\Delta_E^2\omega_r^2}{(\psi_e - C\,r^n)^2}\left[1 \pm \sqrt{\left(1 - \frac{(\psi_e - C\,r^n)^2}{\pi^2\,\Delta_E^2}\right)}\right],$$

whence, after making use of (3.46), we obtain

$$\Omega = \omega - \omega_r\sqrt{\left\{\frac{2\alpha^2}{\left(1 - \frac{r^n}{r_0^n}\right)^2}\left[1 \pm \sqrt{\left(1 - \frac{\left(1 - \frac{r^n}{r_0^n}\right)^2}{\alpha^2}\right)}\right] - 1\right\}}. \tag{3.47}$$

Here we have introduced the definitions

$$\frac{\pi\,\Delta_E}{\psi_e} = \alpha, \tag{3.48}$$

$$\sqrt[n]{\frac{\psi_e}{C}} = r_0. \tag{3.49}$$

Formula (3.47) is valid only when $\Omega < \omega$, i.e. only beyond the critical speed of rotation of the shaft.

Considering now Formula (3.47), we shall attempt first of all to find the boundaries of the region of stability. On these boundaries $r \to 0$, and thus the frequency of vibration is given by

$$\Omega = \omega - \omega_r\sqrt{\left\{2\alpha^2\left[1 \pm \sqrt{\left(1 - \frac{1}{\alpha^2}\right)}\right] - 1\right\}}.$$

On the other hand, it follows from Equations (3.33) that as $r \to 0$, $\Omega \to \Omega_0$. Thus the critical frequencies are

$$\omega_* = \Omega_0 + \omega_r\sqrt{\left\{2\alpha^2\left[1 \pm \sqrt{\left(1 - \frac{1}{\alpha^2}\right)}\right] - 1\right\}}. \tag{3.50}$$

Formula (3.50) gives in general two critical frequencies ω_*^{I} and ω_*^{II}, and as a result of the inequality $\omega_r \ll \Omega_0$ they are both of the order of the "classical" critical velocity Ω_0. In the interval between these velocities the undisturbed motion of the shaft is unstable and outside them it is stable. Thus there is a region of instability lying completely beyond the velocity Ω_0 (Fig. 78).

If $\alpha < 1$, i.e. if $\pi \Delta_E < \psi_e$, Formula (3.50) does not give real values for ω_*, i.e. the region of stability disappears. This case evidently

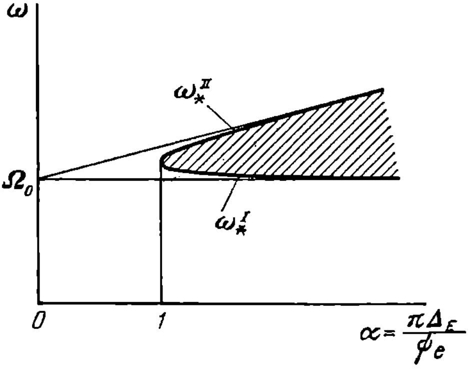


Fig. 78

corresponds to a sufficiently large external friction. If however $\alpha^2 \gg 1$, then Formula (3.50) can be replaced by the simpler expressions

$$\omega_*^{\mathrm{I}} \approx \Omega_0, \quad \omega_*^{\mathrm{II}} \approx \Omega_0 + 2\alpha\,\omega_r.$$

The two possible cases are shown in Fig. 79, where the total dissipation of energy

$$\psi = \psi_e + \psi_i \operatorname{sgn}(\omega - \Omega_0)$$

as $r \to 0$ is measured along the vertical axis. These cases are also illustrated in Fig. 80 which shows the relation between the characteristic exponents s and the angular velocity ω.

Let us now determine the amplitude of the vibrations. We first establish the meaning of the constant r_0 given by (3.49). Suppose that $\omega - \Omega_0 \gg \omega_r$, which corresponds to very high post-critical velocities; Expression (3.45) then becomes

$$\psi_i \approx C r^n.$$

Thus, making use of Equation (3.32), we find that

$$r = \sqrt[n]{\frac{\psi_e}{C}} = r_0, \tag{3.51}$$

i.e. r_0 is the amplitude of vibrations at extremely high post-critical velocities.

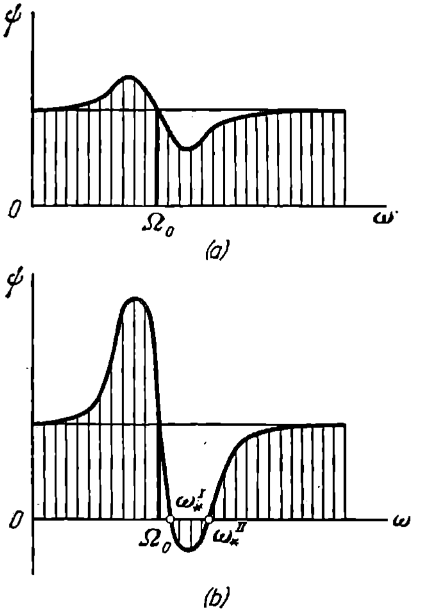


Fig. 79

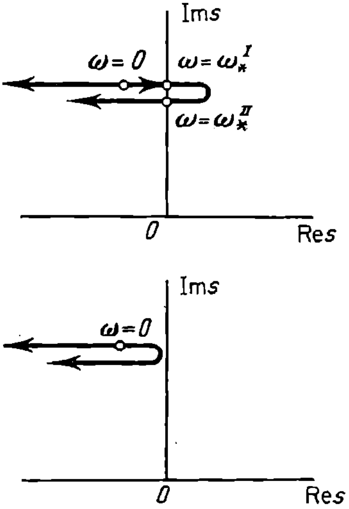


Fig. 80

Formula (3.47) gives real values for Ω if

$$\frac{r^n}{r_0^n} - 1 \leqq \alpha.$$

The amplitude

$$r_{***} = r_0 \sqrt[n]{(1 + \alpha)}$$

is the greatest possible amplitude of vibration (for steady asynchronous precession, to which the solution under consideration corresponds).

In order to establish the relation between amplitude and angular velocity ω, we must find a compatible solution to Equations (3.33) and (3.34). This is most easily carried out graphically, by finding the points of intersection of the curves

$$\Omega - \omega = -\omega_r \sqrt{\left\{\frac{2\alpha^2}{\left(1 - \frac{r^n}{r_0^n}\right)^2}\left[1 \pm \sqrt{\left(1 - \frac{\left(1 - \frac{r^n}{r_0^n}\right)^2}{\alpha^2}\right)}\right] - 1\right\}},$$

$$\Omega - \omega = \Omega_0 \sqrt{[(1 + f(r)]} - \omega.$$

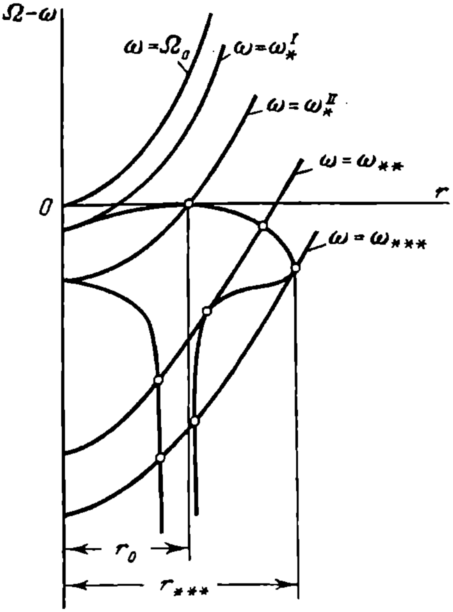


Fig. 81

The construction for the case when $\alpha > 1$ is shown in Fig. 81. The system of equations can have as many as five real roots. In this case the angular velocities ω_*, ω_{**} and ω_{***} correspond to a change in the nature of the vibrations.

The amplitude curve is shown in Fig. 82. As before, the stable branches are indicated by thick lines (the division of the solutions here into stable and unstable solutions has been carried out not by an exact analysis, but on the basis of bifurcation concepts).

Depending on the direction of the change in angular velocity (acceleration or deceleration) and also on the magnitude and nature of the disturbances, different experimental curves, as shown in

Fig. 83, are possible. In spite of all their differences one thing is clear: at low post-critical velocities we have a relation of the type (3.39), and at high post-critical velocities the amplitudes of steady

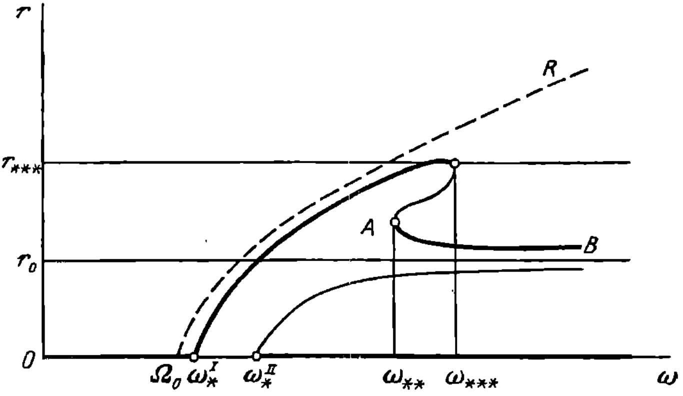


Fig. 82

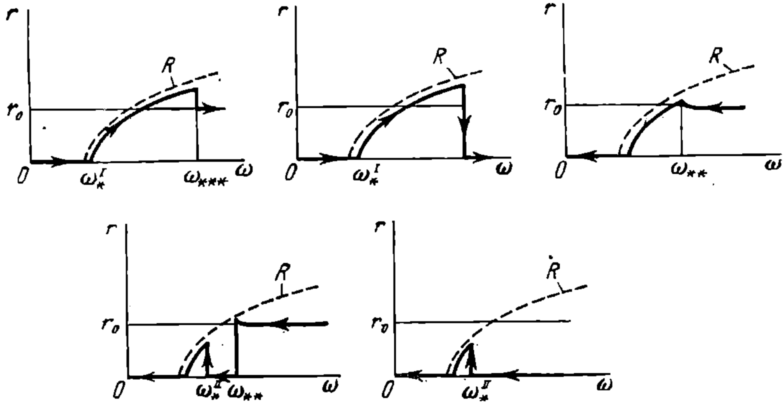


Fig. 83

vibrations are reasonably constant and have a value approximately equal to (3.51).

If $\alpha < 1$ the undisturbed motion is stable "in the small" at all speeds of rotation. In order to set up steady vibrations the system must be subjected to a sufficiently large disturbance. This case can also be represented by several types of experimental diagrams.

3.11. EFFECT OF FRICTIONAL FORCES IN THE CASE OF COMPONENTS SHRUNK ON A SHAFT

So far we have considered the case when internal friction depends on amplitude according to a very simple power law. We could consider the more general case, assuming an arbitrary dependence on amplitude both of internal and external losses. The form of resonance curves varies according to the assumed law governing frictional forces. In particular, the number of stable solutions of the type *AB* in Fig. 82 can vary (if external friction increases with amplitude more rapidly than internal friction, then such solutions, in general, do not exist).

It is well-known that the most pronounced cases of instability are observed when a disk is shrunk on a shaft, and particularly when the disk is mechanically fastened onto the shaft. The frictional forces produced at the contact surface between the bore and the shaft have the same effect as internal frictional forces within the material, except that in this case the effect is more severe.

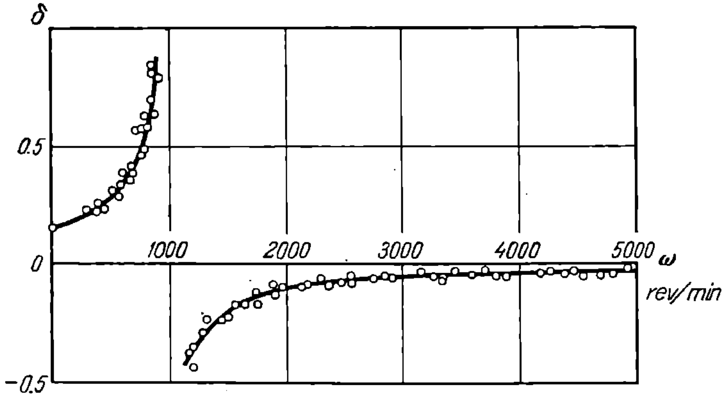


Fig. 84

Frictional forces depend on the state of the contact surface, and in general, as the shaft bends, they increase in magnitude. However, it appears that what is more important is the effect of the relative velocity between the bore and the shaft on the frictional forces, i.e. ultimately the effect of vibration frequency in a rotating system of coordinates. The existence of these effects has been confirmed by experiments carried out by Pozniak at the MEI, (Moscow Energetics Institute). Figure 84 shows the experimental relation between the apparent decrement (or increment) of damping and angular velocity derived for a shaft with two bearings and with a disk attached by means of two fasteners. Up to the critical velocity of approximately 900 rev/min the external friction and the "rotating" friction in the fasteners add, and beyond the critical velocity they subtract. The assumed relation follows curves *a* and *b* in Fig. 85, although in principle a relation of the type *c* is possible. For the falling part of any one of the curves we take the expression

$$\psi_i(r|\omega - \Omega|) = \frac{A(r)}{1 + \dfrac{(\omega - \Omega)^n}{\omega_z^n}}, \tag{3.52}$$

where ω_z and n are constants, $A(r)$ is some function of the displacement.

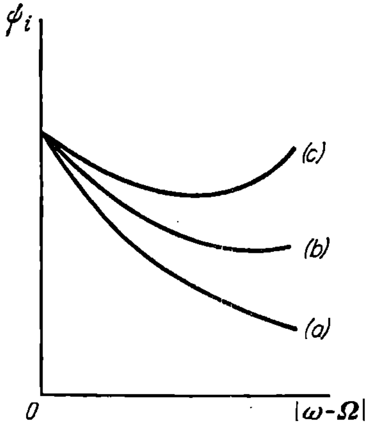


Fig. 85

From Equation (3.32)

$$\psi_i(r, |\omega - \Omega|) = \psi_e(r)$$

we easily obtain

$$\Omega = \omega - \omega_z \sqrt[n]{\left(\frac{A(r)}{\psi_e(r)} - 1\right)}. \tag{3.53}$$

We shall also assume that the friction in the fastener increases rapidly for small amplitudes, but that its increase becomes less when the amplitudes become sufficiently large. Conversely, the increase in external resistances becomes significant only for large amplitudes.

Figure 86 shows the amplitude curve for this case. Instability of the rectilinear form of the shaft occurs immediately after the classical critical velocity Ω is reached, and in this region self-oscillations of small amplitude are induced, which add together to give synchronous precession with amplitude $r = R$. With further increase in angular velocity a cross-over occurs onto the other branch corresponding to steady asynchronous precession. Beyond the velocity ω_{***}, the rectilinear form of the shaft is stable both "in the small" and "in the large". Within the interval $\omega_* < \omega < \omega_{**}$ the rectilinear form of the bar, although stable "in the small", is unstable with respect to fairly large disturbances.

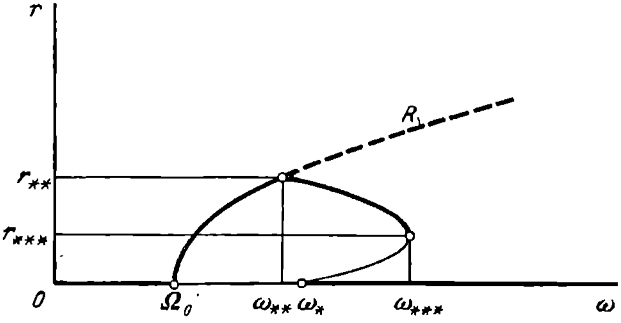


Fig. 86

There is undoubtedly a great need for further theoretical and particularly experimental research into the behavior of shafts rotating at speeds beyond the critical. The results given here should contribute to a fuller understanding of the phenomena associated with the post-critical region, and assist in the design and interpretation of experimental work. In addition, further theoretical research is required into the various laws governing frictional forces, into multi-disk and gyroscopic systems and into quasi-periodic regimes and their stability, etc.

3.12. INSTABILITY OF ROTORS DUE TO THE EFFECT OF AN OIL LAYER IN THE BEARINGS

Phenomena very similar to self-oscillations caused by internal friction in the material of a shaft can result from frictional forces in the layer of oil in its bearings. A wide range of publications exists on this subject, and much has been written in recent years due to the importance of these phenomena in relation to modern high-speed machines.[1]

Consider a rigid rotor of diameter $2R$ rotating at a constant angular velocity ω in a bearing with an initial clearance $\Delta \ll R$ filled with oil (Fig. 87). The oscillatory properties of such a rotor result from the elasticity of the oil film. While the rotor rotates coaxially with the bearing the frictional forces in the bearing reduce to a couple of forces acting in a direction opposite to motion. Suppose now that the center of the rotor O, is displaced relative to the center of the bearing O through a distance $r = \sqrt{(u^2 + r^2)} < \Delta$ and performs precessional motion at an angular velocity Ω. After the center of the rotor is displaced, the friction is no longer uniform over the circumference; it will be larger on the side where the gap is reduced and less where it is increased (Fig. 88). Thus a frictional force will be induced in a direction perpendicular to the displacement r, and to a linear approximation proportional to this displacement. Under certain conditions friction can induce self-oscillations analogous to the oscillations of a shaft resulting from hysteresis.

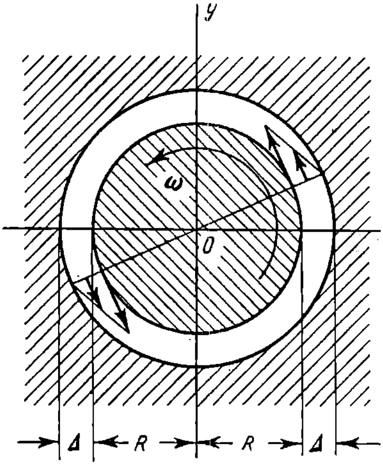


Fig. 87

A formal description of self-oscillations can be given based on the concept of "rotational" friction. The velocity distribution in the film, which can be taken as linear to a high degree of accuracy, will be replaced by a uniform distribution with a mean velocity $\omega R/2$. We transform the equations of oscillation of the rotor

---

[1] See reference on p. 142.

$$\frac{d^2u}{dt^2} + \Omega_0^2 u = 0, \qquad \frac{d^2v}{dt^2} + \Omega_0^2 v = 0$$

to a system of coordinates rotating at the mean velocity $\omega/2$:

$$\frac{d^2\xi}{dt^2} - \omega\frac{d\eta}{dt} + \left(\Omega_0^2 - \frac{\omega^2}{4}\right)\xi = 0,$$

$$\frac{d^2\eta}{dt^2} - \omega\frac{d\xi}{dt} + \left(\Omega_0^2 - \frac{\omega^2}{4}\right)\eta = 0.$$

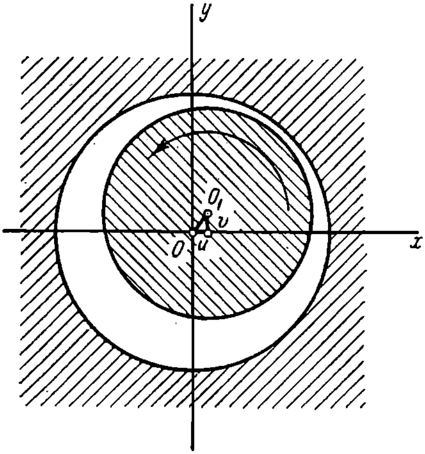


Fig. 88

Introducing into these equations the terms $k_\Delta\, d\xi/dt$ and $k_\Delta\, d\eta/dt$, which take into account viscous friction in the film, returning to a fixed coordinate system and adding terms which take into account external ("non-rotational") resistance, we obtain

$$\left.\begin{aligned}\frac{d^2u}{dt^2} + \left(\frac{\Omega_0^2}{2\pi}\frac{\psi_e}{\Omega} + k_\Delta\right)\frac{du}{dt} + \Omega_0^2 u + \frac{k_\Delta\,\omega}{2}v = 0,\\ \frac{d^2v}{dt^2} + \left(\frac{\Omega_0^2}{2\pi}\frac{\psi_e}{\Omega} + k_\Delta\right)\frac{dv}{dt} + \Omega_0^2 v - \frac{k_\Delta\,\omega}{2}u = 0.\end{aligned}\right\} \quad (3.54)$$

We shall try to find a solution to this system of equations in the form (3.20), where, as before, $w = u + i\,v$. On the boundary of the region of stability

$$\Omega_0^2 - \omega^2 = 0, \qquad \frac{\Omega_0^2 \psi_e}{2\pi} - k_\Delta\left(\frac{\omega}{2} - \Omega\right) = 0. \tag{3.55}$$

Consequently, the critical speed of rotation of the rotor is

$$\omega_* = 2\Omega_0\left(1 + \frac{\Omega_0 \psi_e}{\pi k_\Delta}\right). \tag{3.56}$$

If the external resistance is small, $\omega_* \approx 2\Omega_0$.

In spite of the assumptions made, the foregoing theory gives results which agree satisfactorily with experiment. Figure 89 is based on the experimental results obtained by Newkirk and Lewis[1], and shows the variation of the non-dimensional critical velocity with change in the viscosity coefficient of the oil μ, expressed in centipoises. The results refer to shafts of various ratios of diameter to bearing length and to various values of pressure in the bearing. It can be seen from the graph that the critical velocity ω_* does not exceed twice the value of the "classical" velocity and approaches this value when the viscosity of the film decreases. This is in full agreement with Formula (3.56), which can also be written in the form

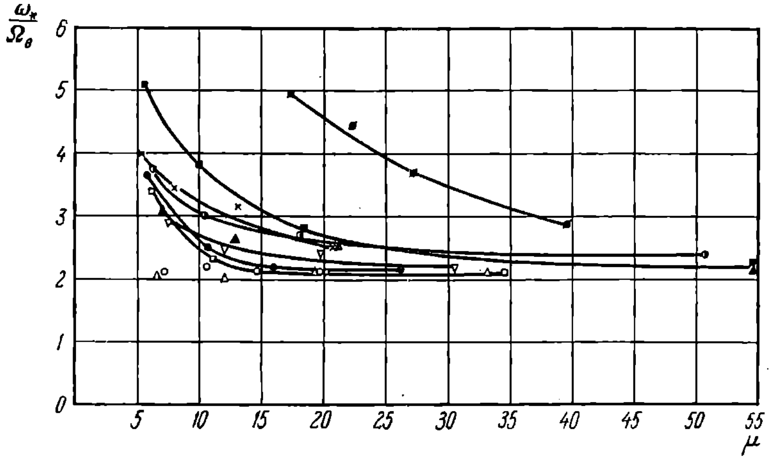


Fig. 89

$$\omega_* = 2\Omega_0\left(1 + \frac{\text{const}}{\mu}\right).$$

---

[1] See reference on p. 142.

Figure 90 shows the recently published results of another investigation[1]. The first amplitude maximum corresponds to forced oscillations caused by eccentricity. The frequency of these oscillations coincides with the angular velocity of rotation. Starting approximately from $\omega = 2\Omega_0$, vibrations are set up due to the effect of the oil film. The frequency of these vibrations remains approximately constant and equal to the natural frequency of oscillation of the rotor over a wide range. This is an indication of the self-oscillatory nature of the phenomenon and agrees closely with theoretical results.

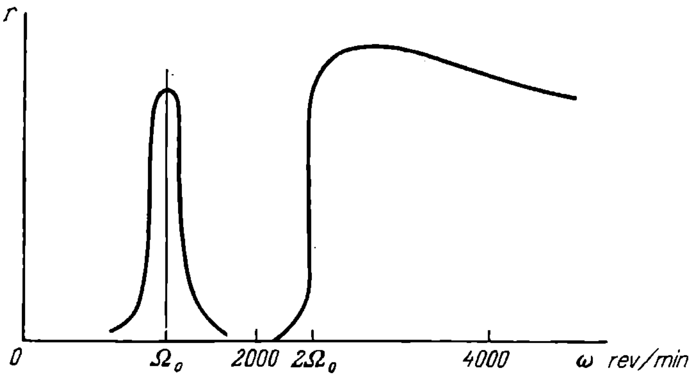


Fig. 90

A more exact theory of instability resulting from friction in an oil layer is based on a consideration of hydrodynamic effects within the layer. The usual assumptions of the hydrodynamic theory of lubrication are made, namely that the thickness of the oil layer is sufficiently small for the effect of curvature and pressure changes within its thickness to be ignored; that the forces due to inertia of the oil are negligibly small compared with viscous forces, and that in the bearing the flow is planar. When the liquid completely fills the gap between the journal and the bearing, the radial and tangential components of the frictional force are given by the formulae

$$P_r = -\frac{12\pi\mu R^3}{\Delta^3}\frac{V_r}{(1-c^2)^{3/2}},$$

$$P_\theta = \frac{12\pi\mu R^2}{\Delta^2}\frac{c}{(2+c^2)(1-c^2)^{1/2}}\left(U - \frac{2R}{r}V_\theta\right).$$

---

[1] Hori, Y. A theory of oil whip. *J. Appl. Mech.*, **26**, No. 2 (1959).

Here $U = \omega R$ is the rotational speed of the rotor, $V_\theta = r\Omega$ is the speed of precessional motion, $V_r = dr/dt$, b is the width of the bearing and $c = r/\Delta$ is the relative eccentricity. If $c \ll 1$ these formulae can be written in the form

$$P_r \approx -K\frac{dr}{dt}, \quad P_\theta \approx K\left(\frac{\omega}{2} - \Omega\right) r, \tag{3.57}$$

where

$$K = \frac{12\pi\mu R^3}{\Delta^3}.$$

We see from the second of Formulae (3.57) that the tangential component P_θ acts in the direction of motion provided $\omega/2 - \Omega > 0$. We therefore conclude that the critical angular velocity cannot be less than twice the natural frequency of oscillation.

It is quite likely that self-oscillations of the rotor are accompanied by separation of the oil layer. Stability of rotational motion under the conditions that the oil only partially fills the gap and that there is an initial eccentricity r_0 has been investigated by Hori and described in his paper. It should be pointed out, however, that theoretical investigations give inconsistent and even contradictory results.[1] There is considerable scope for refinement of the hydrodynamic aspect of this problem if the problem of self-oscillations of rotors caused by oil layers is to be successfully solved.[2]

3.13. INSTABILITY IN CENTRIFUGES INCOMPLETELY FILLED WITH LIQUID

It is wellknown[3] that centrifuges containing a small amount of liquid can undergo dangerous oscillations, the amplitudes of which exceed considerably the amplitudes of oscillation of the empty or completely filled centrifuge. These oscillations take place at frequencies close to the natural frequencies of the centrifuge, and are of a distinctly self-oscillatory nature. The energy absorbed in maintaining the oscillation is provided by the motor; the energy is transmitted by frictional forces in the layer of liquid in the centrifuge. We see here a complete analogy with the effect of internal friction in a material or of friction in the oil layer in bearings.

---

[1] COLE, J. E. Extent of the oil film and shaft vibrations in a solid bearing. *Collection "Machine design"*, No. 3, IL 1958.

[2] We mention the following publications, which appeared during the past years: HOLMES, R. The vibration of a rigid shaft on short sleeve bearings, *J. Mech. Engng. Sci.*, 2, No. 4 (1960); KESTENS, J. *Stabilité de la position de l'arbre dans un palier a graissage hydrodynamique*, Wear, 3, No. 5 (1960); FLOBERG, L. Attitude-eccentricity curves and stability conditions of the infinite journal bearing, *Chalmers tekn. högskol. hande*, No. 235 (1961); See also the book: KAMERON, A. *Theory of lubrication in Engineering*, Mashgiz, 1962. The hydrodynamic phenomena, taking place in the layer, are considered in the paper: POZNIAK, E. L. Dynamic properties of the oil layer in slide bearings, *Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie*, No. 6 (1961).

[3] See the paper by EPISHEV, L. V. referred to on p. 142.

The concept of "rotational" friction can also be successfully applied to the present problem. Suppose that the length of the rotor

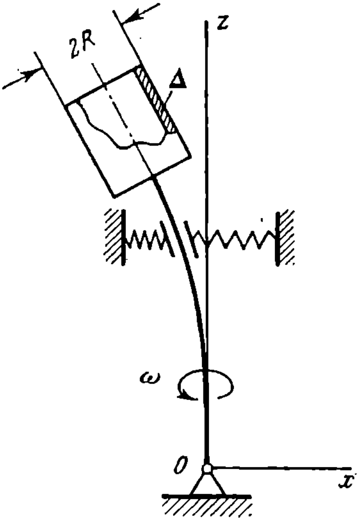


Fig. 91

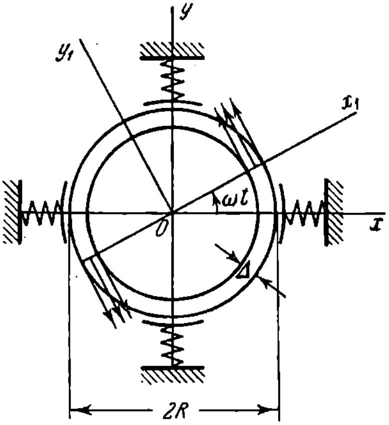


Fig. 92

of a centrifuge rotating at a constant angular velocity ω is small compared with the distance from the fixed support (Fig. 91). The problem can then be reduced to an investigation of the plane motion of the rotor with elastic supports. We shall assume that the mass of the liquid is negligibly small compared with the mass of the rotor, and that the thickness of the layer $\Delta \ll R$ (Fig. 92). We can make an approximation by assuming that in the undisturbed state of motion the whole of the liquid filling the rotor rotates at an angular velocity ω. In the case of small disturbances with components u and v, the rotor will be subjected to frictional forces which depend on the velocity of the rotor relative to the rotating liquid. These forces must be introduced in a coordinate system rotating at an angular velocity ω. The equations of disturbed motion are

$$\left.\begin{aligned} \frac{d^2u}{dt^2} + \left(\frac{\Omega_0^2}{2\pi}\frac{\psi_e}{\Omega} + k_\Delta\right)\frac{du}{dt} + k_\Delta\,\omega\, v + \Omega_0^2 u = 0, \\ \frac{d^2v}{dt^2} + \left(\frac{\Omega_0^2}{2\pi}\frac{\psi_e}{\Omega} + k_\Delta\right)\frac{dv}{dt} - k_\Delta\,\omega\, u + \Omega_0^2 v = 0, \end{aligned}\right\} \tag{3.58}$$

from which we find that the critical velocity is

$$\omega_* = \Omega_0\left(1 + \frac{\Omega_0\,\psi_e}{2\pi\,k_\Delta}\right), \tag{3.59}$$

i.e. instability caused by a film of liquid can occur only beyond the "classical" critical velocity Ω_0.

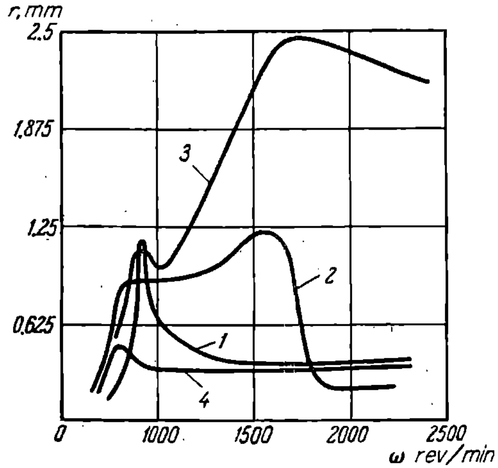


Fig. 93

Figure 93 gives the results obtained from a model centrifuge with a rotor of approximately 1600 cm³ capacity. Curve 1 corresponds to an empty rotor, and its maximum to oscillations caused by eccentricity near $\omega = \Omega_0$. Curve 2 corresponds to the addition of 50 cm³ of liquid to the rotor and curve 3, to the addition of 600 cm³ of liquid. Oscillation commences immediately after the velocity Ω_0 is reached, and is maintained over a wide range of angular velocities, the frequency being close to the natural frequency. When the rotor is completely filled with liquid (curve 4), the oscillations are considerably reduced.

From theoretical considerations we might expect that by putting baffles in the rotor, designed to restrict movement of the liquid, we could make centrifuges run more smoothly. This has been confirmed by experiment. The incorporation of four solid radial baffles totally eliminates self-oscillation. Even if these baffles are fairly shallow and do not reach the free surface, the centrifuge runs more smoothly.

Self-oscillations do not occur in the rotors of separators, the plates of which are immersed in liquid. If, however, filling of the separator takes place while it is rotating at a speed beyond the critical, then at the moment just before the liquid reaches the plates, conditions are prevalent which are likely to give rise to dangerous oscillations.

It will be noted that the stability of rotating, (or in particular gyroscopic), systems containing a liquid with a free surface is of interest from the point of view of a number of other applications. A series of experiments have been carried out in this field[1]. The usual approach is based on the assumption that the liquid is non-viscous and that the disturbances are small. Starting from this assumption and expanding the vector of the displacements of particles of liquid into series in a complete system of functions, we can obtain an infinite system of ordinary differential equations containing pseudo-gyroscopic terms (i.e. terms with generalized coordinates the matrix of which is antisymmetric). The presence of these terms makes the present similar to other problems considered in this book.

3.14. INSTABILITY OF ROTORS IN A MAGNETIC FIELD

A further type of non-conservative problem of elastic stability concerns the action of magnetic and electrodynamic forces on rotating rotors[2]. As a simple example, consider the motion of a flexible shaft carrying a ferromagnetic core situated in a constant magnetic field (Fig. 94). In addition to forces of magnetic attraction, the core is subjected to forces produced by the motion of the core relative to the field. These forces are associated primarily with losses in magnetic hysteresis and Foucault currents. If the rotor and the magnetic field are not rotating, then these forces assist in damping the natural oscillations of the rotor. However, if the rotor rotates relative to the field at a constant velocity, then under certain conditions these forces can cause a sudden transfer of the energy of the field or motor from a purely rotational form to an oscillatory form of motion, and in this way induce self-oscillations of the rotor.

---

[1] See, for example, NARIMANOV, G. S. On the motion of a symmetrical gyroscope partially filled with liquid. *PMM*, **21**, No. 5 (1957). STEWARTSON, K. On the stability of a spinning top containing liquid. *J. Fluid. Mech.* **5**, No. 4 (1959) and others.

[2] See the paper by POZNIAK, E. L. referred to on p. 142.

In order to explain this phenomenon, we make use, as before, of the concept of "rotational" friction. Let ω be the speed of rotation of the rotor and ω_π the speed of rotation of the field. In the same way that internal frictional forces were introduced in a coordinate system rotating at a speed ω, the forces of magnetic friction will be introduced in a system of coordinates rotating at a speed ω_π. After the usual transformations, we arrive at a system of equations analogous to (3.18):

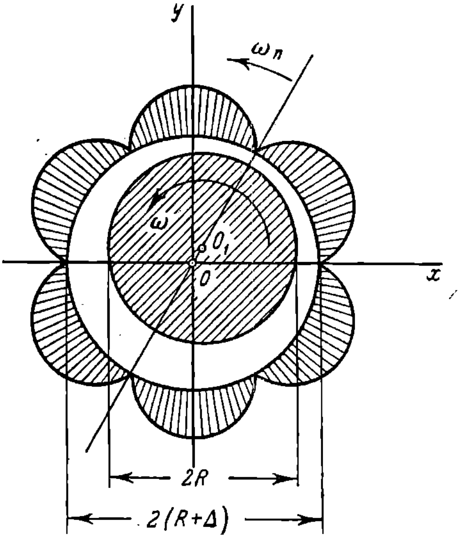


Fig. 94

$$\left.\begin{aligned}
&\frac{d^2u}{dt^2}+\frac{\Omega_0^2}{2\pi}\left(\frac{\psi_e}{\Omega}+\frac{\psi_i}{|\omega-\Omega|}+\frac{\psi_M}{|\omega_\Pi-\Omega|}\right)\frac{du}{dt}+\Omega_0^2u+\\
&\qquad+\frac{\Omega_0^2}{2\pi}\left(\frac{\psi_i\,\omega}{|\omega-\Omega|}+\frac{\psi_M\,\omega_\Pi}{|\omega_\Pi-\Omega|}\right)v=0;\\
&\frac{d^2v}{dt^2}+\frac{\Omega_0^2}{2\pi}\left(\frac{\psi_e}{\Omega}+\frac{\psi_i}{|\omega-\Omega|}+\frac{\psi_M}{|\omega_\Pi-\Omega|}\right)\frac{dv}{dt}+\Omega_0^2v-\\
&\qquad-\frac{\Omega_0^2}{2\pi}\left(\frac{\psi_i\,\omega}{|\omega-\Omega|}+\frac{\psi_M\,\omega_\Pi}{|\omega_\Pi-\Omega|}\right)u=0.
\end{aligned}\right\}\quad(3.60)$$

Here ψ_e, ψ_i and ψ_M represent the relative dissipation of energy caused by external, internal and magnetic friction respectively, Ω is the precession frequency. Hence we obtain a stability condition of the type (3.21), namely

$$\psi_e - \psi_i \operatorname{sgn}(\omega - \Omega_0) - \psi_M \operatorname{sgn}(\omega_\Pi - \Omega_0) > 0. \qquad (3.61)$$

Let us consider Condition (3.61) in more detail. If $\omega < \Omega_0$, $\omega_\pi < \Omega_0$, then both internal and magnetic friction assist in stabilizing steady rotation, and the rectilinear form of the shaft is stable. If $\omega < \Omega_0$, $\omega_\pi > \Omega_0$, then instability can be induced by forces of magnetic friction if $\psi_M > \psi_e + \psi_i$. If $\omega > \Omega_0$, $\omega_\pi < \Omega_0$, instability will occur if $\psi_i > \psi_M + \psi_e$. Finally, if $\omega > \Omega_0$, $\omega_\pi > \Omega_0$, instability occurs if $\psi_M + \psi_i > \psi_e$.

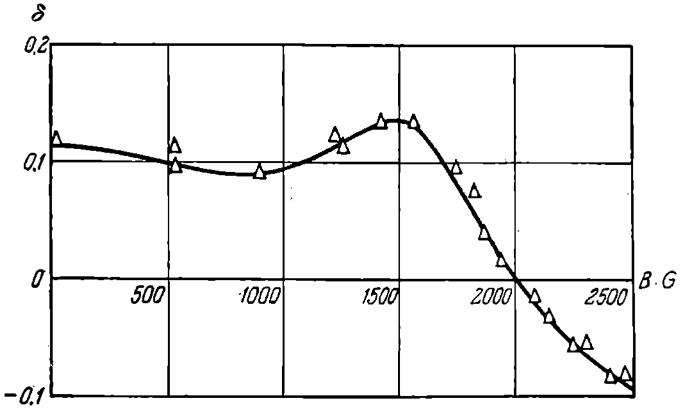


Fig. 95

Figure 95 shows some experimental results relating to the behavior of a flexible shaft with an iron rotor situated in a two-pole field. The magnitude of the maximum induction in the gap, B (in gauss) is measured along the horizontal, and the decrement (or increment) in the small bending oscillations is measured along the vertical. The shaft was fixed to prevent rotation ($\omega = 0$), and the magnetic field was rotated at an angular velocity $\omega_\pi = 3000$ r.p.m. The natural frequency of the rotor in the absence of the field was 1460 cycles/min and fell slightly with increase in induction[1]. Oscillations of the core caused by a lateral impulse are damped if the induction in the gap is small. This implies that the stability condition $\psi_M \approx \psi_e + \psi_i$ is fulfilled. If $B \approx 2000$ gauss, self-oscillations are induced with precession in the direction of rotation of the field; their frequency is approximately 1330 cycles/min, (i.e. equal to the natural frequency in the case of a fixed field).

---

[1] Due to the effect of magnetic attraction.

Similar phenomena are observed not only in the case of rotors carrying a ferromagnetic core, but also in the case of a rotor carrying a coil. Consider the forces acting on a rotating coil in the magnetic field of a stator (Fig. 96). Suppose the induction in the gap is B. Then, considering that the ampere force is proportional to the rate of intersection of lines of force in the field and to the square of the induction, we find that on the conductor of the coil there acts a force

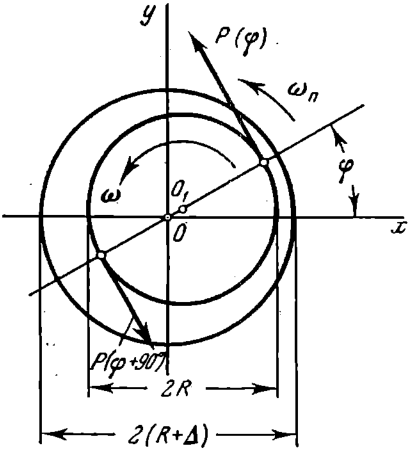


Fig. 96

$$P = \gamma(\omega - \omega_{\Pi})\, B^2,$$

where γ is some coefficient of proportionality. If the rotor is positioned centrally, the forces acting on the coil form a couple which makes a considerable contribution to the turning (or braking) moment. However, if the coil is displaced, the forces are not identical, with the result that a component appears which acts in a direction perpendicular to the plane of the coil.

Assuming that induction is inversely proportional to the air gap Δ, we find that

$$B(\varphi) = B_0 \frac{\Delta}{\Delta - u\cos\varphi - v\sin\varphi}.$$

Thus for small displacements

$$P(\varphi) \approx \gamma(\omega - \omega_\Pi) B_0^2 \left(1 + 2\frac{u}{\Delta}\cos\varphi + 2\frac{v}{\Delta}\sin\varphi\right).$$

The projection of the resultant of the forces applied to the rotor onto fixed coordinate axes can be found from the formulae

$$P_x = \int_0^{2\pi} P(\varphi)\sin\varphi\, c(\varphi)\, d\varphi,$$

$$P_y = \int_0^{2\pi} P(\varphi)\cos\varphi\, c(\varphi)\, d\varphi.$$

Here $c(\varphi)$ is the coil density (the number of turns on the average per radian). In particular, if $c =$ const, $B_0 =$ const, then

$$P_x = -\gamma\pi c(\omega - \omega_\Pi) B_0^2 v = -\gamma_1(\omega - \omega_\Pi) v,$$
$$P_y = \gamma\pi c(\omega - \omega_\Pi) B_0^2 u = \gamma_1(\omega - \omega_\Pi) u.$$

Substitution of these forces in the equations of disturbed motion of the rotor will also lead (under certain conditions) to instability of purely rotational motion.

CHAPTER 4

STABILITY OF ELASTIC BODIES IN A GAS FLOW

1. Short Historical Introduction

One of the most important non-conservative problems of elastic stability is the problem of the stability of elastic bodies situated in a flow of gas or liquid. The forces exerted on the body by the flow depend on the elastic deformation of the body, and for this reason the solution of problems such as these requires the use of the equations both of the theory of elasticity and of the theory of aerodynamics. This has resulted in the growth of a new branch of mechanics—the theory of aeroelasticity—which has led to the development of effective analytical and numerical methods, and which now possesses a wide and ever expanding literature.

The problems of aeroelasticity are of interest mainly in the fields of aeronautical and rocket engineering and to a lesser extent in civil engineering, power generation, etc. It is to be expected that the first problems of this sort were encountered and formulated from the needs of aeronautical engineering. Aeroelastic phenomena (wing and tail flutter and divergence) were probably the cause of numerous accidents in the very early stages of aviation development; an accurate understanding and theoretical explanation of these phenomena came very much later. The theory of *wing divergence* (i.e. the torsional form of loss of static stability) was first proposed by Reissner[1]. An analysis of wing flutter in a flow of incompressible fluid based on the so-called "quasi-stationary" theory was derived by Frazer and Duncan[2], Rauscher[3] and Grossman[4]. Another approach based on a more exact definition of aerodynamic forces and forming the basis of the "non-stationary" theory of flutter was proposed by Küssner[1], Keldysh and Lavrent'ev[2] and Theodorsen[3].

---

[1] Reissner, H. Neuere Probleme aus der Flugzeugtechnik. *Z. für Flugtechnik und Motorluftschiffahrt*, Bd. 17, No. 7 (1926).

[2] Frazer, R. A. and Duncan, W. J. The flutter of airplane wings. *Brit. Aeronaut. Res. Couns. Rep. and Mem.*, No. 1155, 1928.

[3] Rauscher, M. Über die Schwingungen freitragender Flügel. *Luftfahrtforschung*. Bd. 4, No. 3 (1929).

[4] Grossman, E. P. Flutter. *Trudy TsAGI*, No. 284 (1937).

Advances in aviation during the second world war, and particularly during the post-war period, gave considerable stimulus to the development of the theory of aeroelasticity. The main aim was to take into account compressibility and to study flutter and divergence at transonic and supersonic velocities. These developments were reflected in the methods of determining aerodynamic forces. As a result of changes in aircraft design the "elastic" aspect of the problem changed (the effect of sweep, chord bending, etc. was considered). Nowadays it is an almost impossible task to list the vast amount of literature on the application of the theory of aeroelasticity to the design of aircraft wings and tail units; a number of books, some devoted almost exclusively to this question, provide some guide to this literature[4].

In recent years a great deal of attention has been given to problems on the stability of plates and shells at high supersonic speeds. These problems are important in connection with the vibration of the skin of modern aircraft (panel flutter). A number of papers have been written on the flutter of flat panels in a supersonic potential flow.[5] The stability of an infinitely long circular cylindrical shell in a flow of gas, the undisturbed velocity of which is in the direction of the generators, has been studied by the author[6]. The flow was assumed to be potential (subsonic and supersonic) and the disturbances were assumed to be small. The same problem, but in a slightly different form, was subsequently investigated by Miles[1].

---

[1] Küssner, H. G. Schwingungen von Flugzeugflügeln. *Luftfahrtforschung*, Bd. **4**, No. 3 (1929).

[2] Keldysh, M. V. and Lavrent'ev, M. A. On the theory of a vibrating wing. *Tekhn. zametki TsAGI*, No. 45 (1935).

[3] Theodorsen, T. General theory of aerodynamic instability and the mechanism of flutter. *NACA Rep.* No. 496 (1935).

[4] Bisplinghoff, R. L., Ashley, H. and Halfman, R. L. *Aeroelasticity* Addison-Wesley, 1955; Fung, Y. C. *Introduction to the Theory of Aeroelasticity* Wiley, 1955.

[5] Goland, M. and Luke, Y. L. An exact solution for two-dimensional linear panel flutter at supersonic speeds. *J. Aeronaut. Sci.*, **21**, No. 4 (1954); Hedgepeth, J. M., Budiansky, B. and Leonard, R. W. Analysis of flutter at supersonic speed. *J. Aeronaut. Sci.*, **21**, No. 7 (1954); Nelson, H. C. and Cunningham, H. J. Theoretical investigation of flutter of two-dimensional flat panels with one surface exposed to supersonic potential flow. *NACA Rep.* No. 1280, 1956.

[6] Bolotin, V. V. Oscillations and stability of an elastic cylindrical shell in compressible gas flow. *Inzh. sborn.*, **24** (1956).

The particular difficulties of flutter problems result from the fact that aerodynamic forces cannot in general be sufficiently simply expressed in terms of the disturbances of the surfaces exposed to the flow. In the case of high supersonic velocities, however, considerable simplifications are possible, which are based on the asymptotic properties of supersonic flow. Starting with the work of Hayes[2], who formulated the conditions of similitude of a hypersonic flow for thin bodies, a number of approximate methods have been suggested for determining aerodynamic forces. The most simple variant is known as the "law of plane sections"[3], or the "piston theory"[4]; it provides a formula relating the local pressure on the body to the normal component of the velocity of the surface at the point considered. Assuming that the disturbances are small, we can linearize this problem and thus reduce many studies of panel flutter to an investigation of the characteristic values of certain ordinary linear differential equations with constant coefficients[5].

In addition, approximate methods have been derived based on the reduction of the elastic system to a finite-dimension system[6]. The question of the range of application of these methods, which is closely linked with the well-known membrane flutter paradox, is discussed

---

1. Miles, J. W. Supersonic flutter of a cylindrical shell. *J. Aeronaut. Sci.*, **24**, No. 2 (1957); **25**, No. 5 (1958).

2. Hayes, W. D. On the hypersonic similitude. *Quart. Appl. Math.*, **5**, No. 1 (1947).

3. Il'yushin, A. A. The law of plane sections at high supersonic velocities. *PMM*, **20**, No. 6 (1956).

4. Lighthill, M. J. Oscillating airfoils at high Mach number. *J. Aeronaut. Sci.*, **20**, No. 6 (1953); Ashley, H. and Zartarian, C. Piston theory—a new aerodynamic tool for the aeroelastician. *J. Aeronaut. Sci.*, **23**, No. 12 (1956).

5. Hedgepeth, J. M. On the flutter of panels at high Mach numbers. *J. Aeronaut. Sci.*, **23**, No. 6 (1956); Movchan, A. A. On the oscillations of a plate moving in a gas. *PMM*, **20**, No. 2 (1956); On the stability of a panel moving in a gas. *PMM*, **21**, No. 2 (1957); Stability of a blade moving in a gas, *PMM*, **21**, No. 5 (1957); Makhortykh, Zh. K. Stability of a multispan panel moving in a gas. *Izv. Akad. Nauk. USSR*, OTN. "Mekhanika i mashinostroenie, No. 2 (1959).

6. Bolotin, V. V. On the stability of a plate in a flow of compressible gas. "*Voprosy prochnosti materialov i konstruktsii*", Izd-vo. Akad. Nauk USSR (1958); Hedgepeth, J. Flutter of rectangular simply-supported panels at high supersonic speeds. *Journ. Aeronaut. Sci.*, **24**, No. 8 (1957).

in a paper by the author[1]. Some experimental results have also been published[2].

The references listed so far have all referred to the linear treatment of flutter. The non-linear treatment is of interest in two respects. In the first place it enables us to evaluate the amplitudes of deflections and stresses when the critical flutter velocity is exceeded, and in this way to decide to what extent this is dangerous. Secondly, it is to be expected that in certain problems (especially with hypersonic flow) non-linearity of aerodynamic forces can introduce new aeroelastic effects not observed at moderate velocities. These aspects of the problem have been considered in a paper by the author[3]. This paper deals in particular with examples of problems in which the aerodynamic forces are able to maintain undamped motion at velocities less than the critical velocities determined from the linear theory. Subsequent papers[4] have investigated flutter of flat and curved panels taking into account geometrical and aerodynamic non-linearity, aerodynamic heating and the interaction of the panel with stiffeners. It has been shown that as a result of considerable geometrical non-linearity steady flutter of panels acting together with stiffeners takes place at moderate amplitudes. In other cases (for example, in the cylindrical bending of a flat panel with free edges) "hard" excitation of flutter can occur on the boundary of the region of stability. Non-linear problems of flutter of flat panels, when geometrical non-linearity alone is taken into account, have also been investigated by Fung[5] and Shen[6].

---

1. BOLOTIN, V. V. On the application of Galerkin's variational method to problems of flutter of elastic panels. *Izv. vyssh. shkoly, ser. "Mashinostroenie"*, No. 11 (1959).

2. MIKISHEV, G. N. Experimental investigation of self-oscillations of a square plate in a supersonic flow. *Izv. Akad. Nauk USSR*, OTN, "Mekhanika i mashinostroenie", No. 1 (1959).

3. BOLOTIN, V. V. On critical velocities in the non-linear theory of aeroelasticity. *Nauchn. dokl. vyssh. shkoly, ser. "Mashinostroenie i priborostroenie"*, No. 3 (1958).

4. BOLOTIN, V. V., GAVRILOV, Iu. V., MAKAROV, B. P. and SHVEIKO, Iu. Iu. Non-linear problems of the stability of flat panels at high supersonic velocities. *Izv. Akad. Nauk USSR*, OTN, "Mekhanika i mashinostroenie", No. 3 (1959); BOLOTIN, V. V. Non-linear flutter of plates and shells. *Inzh. sborn.*, **29** (1960).

5. FUNG, Y. C. On two-dimensional panel flutter, *J. Aeronaut. Sci.*, **25**, No. 3 (1958).

6. SHEN, S. F. An approximate Analysis of Nonlinear flutter problems. *J. Aeronaut. Sci.*, **36**, No. 1 (1959).

4.2. FLUTTER OF A WING AS A NON-CONSERVATIVE. PROBLEM OF ELASTIC STABILITY

In order to show the close relation between problems of flutter of elastic elements in a flow of gas and the non-conservative problems of elastic stability already discussed, let us consider the simple problem of the bending—torsional flutter of a cantilever wing. This problem has been investigated in considerable detail and is dealt with in special texts.[1] For this reason we shall not go too deeply into technical

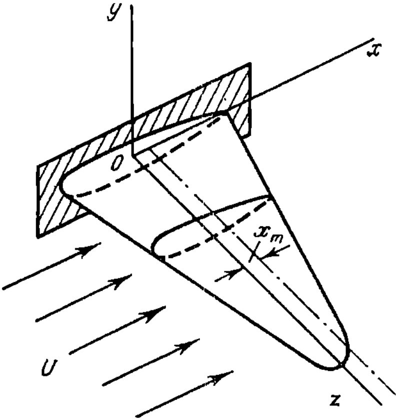


Fig. 97

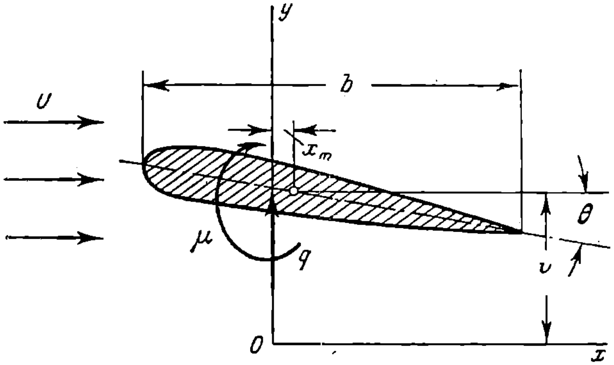


Fig. 98

---

[1] See, for example, reference 4 on p. 200.

details or detailed calculations; instead we shall confine our attention to a comparatively simple form of the problem.

Consider a wing with a straight bending-center axis lying perpendicular to the flow (Fig. 97). We shall treat the wing as an elastic bar, the stiffness of which in the direction of flow is a maximum and very large compared with its other bending stiffness and also with its torsional stiffness. Under these conditions the deformed state of the wing can be defined by the displacements of the centers of bending $v(z, t)$ and by the angles of rotation of lateral sections $\theta(z, t)$. These displacements are shown in Fig. 98. The equations of small oscillations of the wing are

$$\left.\begin{aligned}
&\frac{\partial^2}{\partial z^2}\left(EJ\frac{\partial^2 v}{\partial z^2}\right) + m\frac{\partial^2 v}{\partial t^2} - m x_m \frac{\partial^2 \theta}{\partial t^2} = q(z, t),\\
&\frac{\partial^2}{\partial z^2}\left(EJ_\omega\frac{\partial^2 \theta}{\partial z^2}\right) - \frac{\partial}{\partial z}\left(GJ_d\frac{\partial \theta}{\partial z}\right) - m x_m\frac{\partial^2 v}{\partial t^2} +\\
&\qquad + m r^2\frac{\partial^2 \theta}{\partial t^2} = \mu(z, t).
\end{aligned}\right\} \qquad (4.1)$$

Here EJ, GJ_d and EJ_ω are the bending, torsional and sectorial stiffnesses, respectively, m is the mass per unit length of the wing (along its span), x_m is the distance from the bending center to the mass center, r is the radius of gyration of the section about the bending center, q and μ are the aerodynamic force and aerodynamic moment, respectively, per unit length of the wing.

Certain difficulties are involved in determining the aerodynamic forces acting on an oscillating wing. One simplification which is often introduced is to calculate the aerodynamic forces for the case of plane flow. If the wing is long enough and if its parameter changes only slightly along its span, then this approximation is reasonable. Then in the linear approximation

$$\left.\begin{aligned}
q(z, t) &= A_{12}\theta + B_{11}\frac{\partial v}{\partial t} + B_{12}\frac{\partial \theta}{\partial t},\\
\mu(z, t) &= A_{21} v + A_{22}\theta + B_{21}\frac{\partial v}{\partial t} + B_{22}\frac{\partial \theta}{\partial t}.
\end{aligned}\right\} \qquad (4.2)$$

Here A_{ik} and B_{ik} are coefficients which, in general, depend on the velocity of the approaching flow U and in general on the coordinate z as well. In addition, in some cases they also depend on the nature of the motion under investigation. For example, if the motion is periodic with frequency Ω, then the coefficients in Expressions (4.2) can also depend on the non-dimensional parameter, the reduced frequency $\Omega\, b/U$ (here b is the chord length).

We shall not bother to give the explicit expressions for the coefficients A_{ik} and B_{ik}, simply observing that they are of a very simple form in two cases: if they are calculated according to the "quasi-stationary theory", and also if high supersonic velocities are considered and the "piston theory" is used.

Consider first the stationary solution of the system of Equations (4.1). Then $v \equiv 0$ and the second equation can be written in the form

$$\frac{d^2}{dz^2}\left(E J_\omega \frac{d^2\theta}{dz^2}\right) - \frac{d}{dz}\left(G J_d \frac{d\theta}{dz}\right) - \frac{\partial c_\mu}{\partial \theta}\frac{\varrho\, U^2 b^2}{2}\theta = 0. \qquad (4.3)$$

Here the coefficient A_{22} has been expressed in terms of the empirical value of the derivative of the coefficient of the aerodynamic moment c_μ; ϱ is the density of the approaching flow. For a cantilever wing the boundary conditions are

$$\theta(0) = \frac{d\theta(0)}{dz} = \frac{d^2\theta(l)}{dz^2} = \frac{d^3\theta(l)}{dz^3} = 0. \qquad (4.4)$$

If $U = 0$ the boundary-value problem has no solutions other than the trivial solution $\theta = 0$. The minimum value of the velocity U for which the boundary-value problem has a non-trivial solution corresponds to bifurcation of the forms of equilibrium; in addition to the trivial (untwisted) form, there appears another very close form of equilibrium which is accompanied by twisting of the wing. An investigation of the appropriate non-homogeneous problem shows that as the velocity indicated is approached, a sharp increase in the angle of twist, or divergence (overtwist) of the wing, occurs. Wing divergence is completely analogous to normal static instability in elastic systems, and the method of finding the critical divergence velocity is completely analogous to *Euler's method* in the theory of elastic stability.

In the case of a wing of constant section with zero sectorial stiffness, the problem is considerably simplified:

$$\frac{d^2\theta}{dz^2} + k^2\theta = 0, \qquad \theta(0) = \frac{d\theta(0)}{dz} = 0,$$

where

$$k^2 = \frac{\varrho\, U^2 b^2}{2G\, J_d}\,\frac{\partial c_\mu}{\partial \theta}.$$

Non-trivial solutions exist of $\cos k l = 0$; thus the critical velocity of divergence is

$$U_* = \frac{\pi}{2lb}\sqrt{\frac{2GJ_d}{\varrho \frac{\partial c_\mu}{\partial \theta}}}. \tag{4.5}$$

Consider now the problem of wing flutter, referring again to the system of Equations (4.1), putting

$$v(z, t) = V(z)\, e^{st},$$
$$\theta(z, t) = \Theta(z)\, e^{st},$$

where $V(z)$ and $\Theta(z)$ are certain functions of the mode of oscillations and s is a characteristic exponent. Substitution leads to the equations

$$\left.\begin{aligned} &\frac{d^2}{dz^2}\left(EJ\frac{d^2 V}{dz^2}\right) + m s^2 V - m x_m s^2 \Theta = \\ &\qquad = A_{12}\Theta + B_{11} s V + B_{12} s \Theta, \\ &\frac{d^2}{dz^2}\left(EJ_\omega \frac{d^2\Theta}{dz^2}\right) - \frac{d}{dz}\left(GJ_d\frac{d\Theta}{dz}\right) - m x_m s^2 V + m r^2 s^2 \Theta = \\ &\qquad = (A_{21} + B_{21} s) V + (A_{22} + B_{22} s)\Theta \end{aligned}\right\} \tag{4.6}$$

and to boundary conditions of the type (4.4).

The problem consists of investigating the behavior of the exponents in relation to the velocity U. The equilibrium of the wing in a gas flow is stable if all the exponents s lie in the left-hand half-plane of the complex variable. The lowest value of the velocity U at which even one of the exponents crosses into the right-hand half-plane is the critical velocity. Transition into the right-hand half-plane indicates instability of the oscillatory type (flutter). The one exception is when transition takes place through the point $s = 0$; this case obviously corresponds to divergence.

We see then that there is a complete analogy with the non-conservative problems of elastic stability already considered. As before, the coupled interaction of different degrees of freedom is of considerable importance also in the case of bending-torsional flutter. This aspect of the problem becomes clear if we pass from a system with an infinite number of degrees of freedom to a system with a finite number. The usual approach is to assume that the motion of the wing is defined in the following way:

$$v(z, t) = \sum g_k(t)\,\varphi_k(z); \qquad \theta(z, t) = \sum f_k(t)\,\psi_k(z). \tag{4.7}$$

Here g_k and f_k are generalized coordinates, the first of which corresponds to bending, the second to torsion, $\varphi_k(z)$ and $\psi_k(z)$ are functions which satisfy the boundary conditions (for example, "beam" functions). Substituting Expressions (4.7) into Equations (4.6) and using Galerkin's variational method, we arrive at a system of two ordinary differential equations in $g_k(t)$ and $f_k(t)$. These equations have the same structure as Equations (1.79); however, the loading parameter is now replaced by the velocity U.

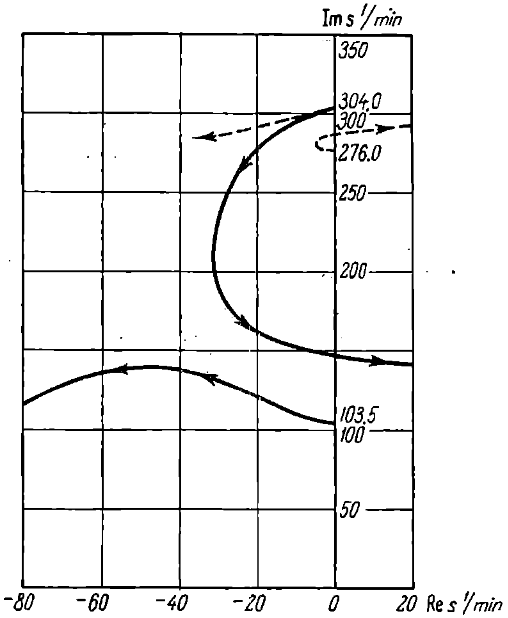


Fig. 99

Comparison shows that if we retain the first term in both of the series (4.7), we obtain perfectly satisfactory results. Bending-torsional flutter is thus maintained by the interaction of the basic modes of bending and torsional oscillations. The relation between the partial natural frequency of bending oscillations Ω_y and the partial natural frequency of torsional oscillations Ω_θ has a considerable effect on the critical flutter velocity, which is to be expected from general considerations. Figure 99 shows the behavior of the characteristic exponents for two different ratios of the partial frequencies[1]. As the ratio Ω_y/Ω_θ increases from 0.34 to 0.91 the critical velocity parameter falls from 740 to 250. Figure 100 shows one example of the way in which the critical velocity varies with the ratios x_n/b and Ω_y/Ω_θ. The magnitude of the critical velocity of divergence is also shown. Depending on the relation between the parameters, the critical flutter velocity can be either smaller or larger than the critical velocity of divergence.

---

[1] Goland, M. and Luke, Y. L. A study of the bending-torsion aeroelastic modes for an airplane wing. *J. Aeronaut. Sci.*, **16**, No. 7 (1949).

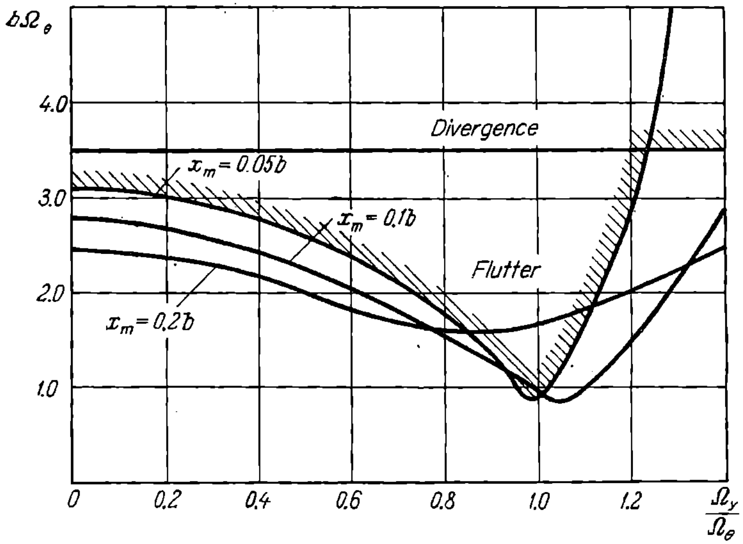


Fig. 100

4.3. General Formulation of Problems of Stability of Elastic Bodies in Potential Gas Flow

Aeroelastic problems can be formulated in general terms with the widest assumptions concerning the properties of the elastic body and the way in which the body interacts with the flow. We shall now consider the case when the flow is potential and continuous everywhere, except perhaps at a finite number of discontinuity surfaces, and when the disturbances are small.

Consider an elastic body of volume V and surface area S situated in a steady potential flow of gas. We take this as a steady state of undisturbed motion and choose a Cartesian system of coordinates x_1, x_2, x_3 connected to the body[1]. We shall assume that the undisturbed velocity has potential $\varphi(x_1, x_2, x_3)$ i.e. that the components of velocity v_1, v_2, v_3 can be expressed in terms of φ in the following manner:

$$v_k = \frac{\partial \varphi}{\partial x_k}.$$

The disturbed state can be defined by the displacement vector of points in the body u_k and by the potential of the disturbances $\bar\varphi$, i.e.

$$\tilde v_k = v_k + \frac{\partial \bar\varphi}{\partial x_k},$$

$$\tilde u_k = u_k$$

(the symbol $\sim$ refers to the disturbed parameters). We shall assume that the disturbances causing the deformations of the body are sufficiently small for their squares and higher powers to be ignored. Then, starting from the equation for the potential flow of a compressible fluid[2],

$$\frac{\partial^2 \varphi}{\partial x_k^2} - \frac{2v_l}{c_\infty^2}\frac{\partial^2 \varphi}{\partial x_l\,\partial t} - \frac{v_l v_k}{c_\infty^2}\frac{\partial^2 \varphi}{\partial x_l\,\partial x_k} - \frac{1}{c_\infty^2}\frac{\partial^2 \varphi}{\partial t^2} = 0,$$

we obtain the following linearized equation for the potential of the disturbed velocities

$$c_\infty^2 \frac{\partial^2 \bar\varphi}{\partial x_k^2} - 2v_l \frac{\partial^2 \bar\varphi}{d x_l\,\partial t} - v_l v_k \frac{\partial^2 \bar\varphi}{\partial x_l\,\partial x_k} - \\ - v_k\left(\frac{\partial v_k}{\partial x_l} + \frac{\partial v_i}{\partial x_k}\right)\frac{\partial \bar\varphi}{\partial x_i} - \frac{\partial^2 \bar\varphi}{\partial t^2} = 0. \qquad (4.8)$$

Here $c_\infty = \sqrt{(\partial p/\partial \varrho)}$ is the velocity of sound forthe undisturbed flow, p is the pressure and ϱ is the density. For example, if the direction of the undisturbed velocity is along the axis of x_1, i.e. if $v_x = U$, $v_y = v_2 = 0$, then, introducing the notation for the Mach number of the undisturbed flow

---

[1] From now on we shall observe the usual convention regarding summation with respect to "dummy" (subscripts) (see page 28).

[2] See, for example, Frankl', F. I. and Karpovich, E. A. *Gas Dynamics of Thin Bodies* (Gazodinamika tonkikh tel). Gostekhizdat, 1948.

$$M = \frac{U}{c_\infty},$$

we can write Equation (4.8) in the form

$$(1 - M^2)\frac{\partial^2 \bar{\varphi}}{\partial x^2} + \frac{\partial^2 \bar{\varphi}}{\partial y^2} - \frac{2M}{c_\infty}\frac{\partial^2 \bar{\varphi}}{\partial x\, \partial z} - \frac{1}{c_\infty^2}\frac{\partial^2 \bar{\varphi}}{\partial t^2} = 0. \tag{4.9}$$

For the elastic displacements u_i we have the system of equations

$$\mathscr{L}_{ik}\, u_k - \varrho_0 \frac{\partial^2 u_i}{\partial t^2} = 0, \tag{4.10}$$

where ϱ_0 is the density of the material of the body. For example, in the case of an isotropic body,

$$\mathscr{L}_{ik} = \mu \frac{\partial^2}{\partial x_j^2}\delta_{ik} + (\lambda + \mu)\frac{\partial^2}{\partial x_i\, \partial x_k},$$

where λ and μ are the Lamé coefficients and δ_{ik} is the Kronecker delta.

Now consider the boundary conditions which we shall split into two categories—general and special conditions. General conditions are those that are satisfied at any point of the contact surface S between the body and the gas. Special boundary conditions take into account the way in which the body is supported, the conditions at infinity etc., and vary from problem to problem. Here it is clearly sufficient to consider the general boundary conditions.

We now impose the condition of impermeability of the surface of the body. Let $F(a_1, a_2, a_3)$ be the equation of the surface of the body in Lagrangian coordinates, and x_1, x_2, x_3 the Eulerian coordinates. We then obtain the following equation for the disturbed surface

$$F(x_1 - u_1,\ x_2 - u_2,\ x_3 - u_3,) = F_1(x_1, x_2, x_3, t) = 0.$$

The material derivative of the function F must be zero (there is no gas flow through the disturbed surface), i.e.

$$\frac{\partial F_1}{\partial t} + \bar{v}_k \frac{\partial F_1}{\partial x_k} = 0.$$

Thus the condition

$$\frac{\partial \varphi}{\partial n} = \left(\frac{\partial u_k}{\partial t} + \bar{v}_i \frac{\partial u_k}{\partial x_i}\right) n_k,$$

where n_k are the components of the vector of the external normal to the surface of the body, must be satisfied over the whole of the surface S. After linearization we obtain

$$\frac{\partial \bar{\varphi}}{\partial n} = \left(\frac{\partial u_k}{\partial t} + \tilde{v}_i \frac{\partial u_k}{\partial x_i}\right) n_k \quad \text{on} \quad S. \tag{4.11}$$

The dynamic condition on the surface S is of the form $\sigma_{ik} n_k = p_i$, where σ_{ik} are the components of the stress tensor within the body, $p_i = pn_i$ are the components of the pressure p. Composing the Cauchy integral for disturbed motion

$$\frac{\partial \tilde{\varphi}}{\partial t} + \frac{1}{2} v_k^2 + \frac{\tilde{p} - p}{\varrho} = F(t),$$

where $F(t)$ is a function of time, we find, after linearization, that

$$\bar{p} = -\varrho\left(\frac{\partial \bar{\varphi}}{\partial t} + v_k \frac{\partial \bar{\varphi}}{\partial x_k}\right). \tag{4.12}$$

Consequently

$$\bar{\sigma}_{ik} n_k = -\varrho\, n_i \left(\frac{\partial \bar{\varphi}}{\partial t} + v_k \frac{\partial \bar{\varphi}}{\partial x_k}\right) \quad \text{on} \quad S. \tag{4.13}$$

By supplementing boundary Conditions (4.11) and (4.13) by the special boundary conditions, we obtain the complete formulation of the problem.

The potential of the disturbances $\bar{\varphi}$ can always be expressed in terms of the normal component $\partial \bar{\varphi} / \partial u$ of the velocity on the surface of the body, as follows:

$$\bar{\varphi}(A) = \iint\limits_S K(A, B) \frac{\partial \bar{\varphi}(B)}{\partial n} dS_B. \tag{4.14}$$

Here $K(A, B)$ is a kernel which depends on the coordinates x_1, x_2, x_3 and ξ_1, ξ_2, ξ_3 of two arbitrary points A and B on the surface of the body. If the fluid is incompressible, $K(A, B)$ is Green's function for Neumann's problem (in an actual construction of this kernel the conditions on discontinuity surfaces would have to be eliminated), In the case of a flat plate exposed to two-dimensional supersonic flow, when the plate lies in the plane of undisturbed flow and oscillates at a frequency Ω, the kernel $K(A, B)$ assumes the form[1]

$$K(A, B) = \begin{cases} \frac{1}{\sqrt{(M^2 - 1)}} \exp\left[- \frac{i M \Omega (x - \xi)}{c_\infty (M^2 - 1)} \right] J_0 \left[\frac{\Omega (x - \xi)}{c_\infty (M^2 - 1)} \right] d\xi \\ \qquad\qquad \text{for } 0 \leqq \xi \leqq x, \\ 0 \quad \text{for } \xi > x, \end{cases} \tag{4.15}$$

where J_0 is a Bessel function of zero order.

By using Relations (4.11) and (4.14), we can eliminate the potential $\bar{\varphi}$:

$$\bar{\varphi}(A) = \iint_S K(A,B) \left[\frac{\partial u_k(B,t)}{\partial t} + v_i \frac{\partial u_k(B,t)}{\partial x_i} \right] n_k \, dS_B .$$

The problem can then be described by the usual Equations (4.10) of the dynamic theory of elasticity with the boundary condition $\sigma_{ik} n_k = p n_i$. Then

$$p = -\varrho \left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \right) \iint_S K(A,B) \left[\frac{\partial u_k(B,t)}{\partial t} + v_j(B) \frac{\partial u_k(B,t)}{\partial x_j} \right] n_k \, dS_B .$$

We express the displacement vector u_i in the form of an expansion in the modes of natural oscillations of an elastic body in vacuo:

$$u_i(x_1, x_2, x_3, t) = \sum f_n(t) \, \psi_{in}(x_1, x_2, x_3).$$

We write the equations of Galerkin's variational method in the form of (1.64):

$$\int_V \left(\frac{\partial \sigma_{ik}}{\partial x_k} - \varrho_0 \frac{\partial^2 u_i}{\partial t^2} \right) \psi_{im} \, dV - \iint_S (\sigma_{ik} n_k - p n_i) \psi_{im} \, dS = 0 \tag{4.16}$$

$$(m = 1, 2, \ldots),$$

and transform the integral

$$\iint_S p n_i \psi_{im} \, dS = -\varrho \iint_S \left(\frac{\partial}{\partial t} + v_i \frac{\partial}{\partial x_i} \right) \left\{ \iint_S K(A,B) \left[\frac{\partial u_k(B,t)}{\partial t} + v_j(B) \frac{\partial u_k(B,t)}{\partial \xi_j} \right] n_k \, dS_B \right\} n_i \psi_{im}(A) \, dS_A =$$

$$= -\varrho \sum \left(a_{mn} \frac{d^2 f_n}{dt^2} + 2 U b_{mn} \frac{d f_n}{dt} + U^2 c_{mn} f_n \right).$$

---

[1] See Section 4.7.

Here

$$a_{mn} = \iint\limits_S \left[\iint\limits_S K(A, B)\, \psi_{kn}(B)\, n_k\, dS_B \right] \psi_{im}(A)\, n_i\, dS_A ,$$

$$b_{mn} = \frac{1}{2U} \iint\limits_S \left\{ \iint\limits_S K(A,B) \left[v_j(B) \frac{\partial \psi_{kn}(B)}{\partial \xi_j} \right] n_k\, dS_B + \right.$$

$$\left. + \, v_i(A) \frac{\partial}{\partial x_i} \iint\limits_S K(A, B)\, \psi_{kn}(B)\, n_k\, dS_B \right\} n_i\, \psi_{im}(A)\, dS_A ,$$

$$c_{mn} = \frac{1}{U^2} \iint\limits_S \left\{ v_k(A) \frac{\partial}{\partial x_i} \iint\limits_S K(A, B)\, v_j(B) \times \right.$$

$$\left. \times \frac{\partial \psi_{kn}(B)}{\partial \xi_j}\, n_k\, dS_B \right\} n_i\, \psi_{im}(A)\, dS_A .$$

Equations (4.16) become

$$\sum \left(\delta_{mn} + \frac{\varrho}{\varrho_0} a_{mn} \right) \frac{d^2 f_n}{dt^2} + \sum \left(\varepsilon_m \delta_{mn} + 2U \frac{\varrho}{\varrho_0} b_{mn} \right) \frac{df_n}{dt} +$$

$$+ \sum \left(\Omega_m^2 \delta_{mn} - U^2 \frac{\varrho}{\varrho_0} c_{mn} \right) f_n = 0 \quad (m = 1, 2, \ldots) . \tag{4.17}$$

Here ϱ, ϱ_0 and U represent certain characteristic values of the density of the gas, of the elastic body and of the undisturbed velocity of flow, respectively, and Ω_m is the natural frequency of oscillations in vacuo. In Equations (4.17) terms containing the matrix $\varepsilon_m \delta_{mn}$ have been introduced, which take into account damping induced by the structure.

From Equations (4.17) we obtain a formulation of the basic problems of small deformations of an elastic body in a potential flow of gas. The equation of the natural undamped oscillations in a gas at rest is given by

$$\left| \Omega_m^2 \delta_{mn} - \left(\delta_{mn} + \frac{\varrho}{\varrho_0} a_{mn} \right) \Omega^2 \right| = 0 , \tag{4.18}$$

from which we see that the matrix a_{mn} characterizes the attached mass of the medium. Bifurcation of the forms of equilibrium (divergence) occurs at a velocity U, which is the smallest root of the equation

$$\left| \Omega_m^2 \delta_{mn} - U^2 \frac{\varrho}{\varrho_0} c_{mn} \right| = 0 . \tag{4.19}$$

Finally, in order to determine the critical flutter velocity we substitute the expressions for $f_m = F_m e^{st}$ in Equations (4.17) and consider the determinant

$$\left| \left(\delta_{mn} + \frac{\varrho}{\varrho_0} a_{mn}\right) s^2 + \left(\varepsilon_m \delta_{mn} + 2U \frac{\varrho}{\varrho_0} b_{mn}\right) s + \right. \left. + \left(\Omega_m^2 \delta_{mn} - U^2 \frac{\varrho}{\varrho_0} c_{mn}\right) \right| = 0. \qquad (4.20)$$

The undisturbed form of equilibrium is stable if all the characteristic exponents s lie in the left-hand half-plane of the complex variable. The smallest value of the velocity U at which one of the characteristic exponents s crosses over into the right-hand half-plane, whilst still remaining complex, is the critical flutter velocity. An actual determination of this velocity might be complicated by the fact that the coefficients a_{mn}, b_{mn} and c_{mn} can depend on non-dimensional combinations of the velocity U and the frequency of oscillations of the body Ω on the boundary of the region of flutter.

4.4. STABILITY OF AN ELASTIC CYLINDRICAL SHELL IN COMPRESSIBLE GAS FLOW

We shall illustrate the foregoing general principles by taking as an example the problem of the stability of a circular cylindrical shell in a flow of gas (Fig. 101). In doing so we shall for the most part follow the procedure outlined in a paper by the author[1]. We postulate a potential flow with velocity of undisturbed flow U_i inside and U_e outside the shell (where $U_i \gtrless U_e$). The radius of the middle surface of the shell will be denoted by R and its thickness by h, and we shall make the assumption that the equations of the bending theory are applicable.

---

[1] BOLOTIN, V. V. Oscillations and stability of an elastic cylindrical shell in a flow of compressible gas. *Inzh. sborn.*, **24** (1956). Further developments may be found in the following works: BOLOTIN, V. V. On the applications of the "law of plane sections" for the determination of aerodynamic forces, acting on vibrating shells. *Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie*, No. 1 (1961); DZYGADLO, Z., KALISKI, S. Self-exited vibrations of a stiffened inelastic shell in a linearized supersonic flow, *Proceedings of vibration problems*, **1**, No. 3 (1960); LIBRESCU, L. Vibratiile structurilor elastic subtiri de forma cilindrica circulara, plasate intr-un curent fluid supersonic, *Studii si cercetari de Mecanica Aplicata*, **12**, No. 1)1961); LIBRESCU, L. Vibrations of non-homogeneous circular cylindrical shells in supersonic fluid flow, *Revue de Mecanique Appliquee*, **6**, No. 5 (1961); LIBRESCU, L. Vibratiile si stabilitatea aeroelasticá a structurilor cilindrice subtiri, neomogene plascte intr-un curent fluid compresibil, *Studii si cercetari de Mecanica Aplicata*, **13**, No. 4 (1962).

We shall employ cylindrical coordinates (x, r, θ), taking the x-axis as the axis of the cylinder. The displacements of the middle surface from the membrane state are defined by the tangential displacements $u(x, \theta)$ and $v(x, \theta)$ and by the normal displacement $w(x, \theta)$. The positive directions of the displacements are shown in Fig. 101.

The equations of motion of the shell are

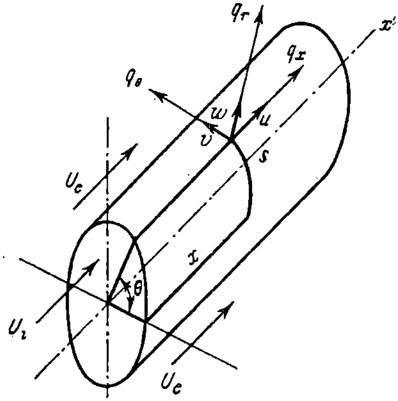


Fig. 101

$$\left.\begin{aligned}
\mathscr{L}_{11} u + \mathscr{L}_{12} v + \mathscr{L}_{13} w &= -\frac{1-\mu^2}{Eh} q_x, \\
\mathscr{L}_{21} u + \mathscr{L}_{22} v + \mathscr{L}_{23} w &= -\frac{1-\mu^2}{Eh} q_\theta, \\
\mathscr{L}_{31} u + \mathscr{L}_{32} v + \mathscr{L}_{33} w &= \frac{1-\mu^2}{Eh} q_r,
\end{aligned}\right\} \qquad (4.21)$$

where $\mathscr{L}_{ik}$ are differential operators defined as follows:

$$\mathscr{L}_{11} = \frac{\partial^2}{\partial x^2} + \frac{1-\mu}{2}\frac{\partial^2}{\partial s^2}; \qquad \mathscr{L}_{22} = \frac{1-\mu}{2}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2};$$

$$\mathscr{L}_{12} = \mathscr{L}_{21} = \frac{1+\mu}{2}\frac{\partial^2}{\partial x \partial s};$$

$$\mathscr{L}_{23} = \mathscr{L}_{32} = \frac{1}{R}\frac{\partial}{\partial s} - \frac{3-\mu}{2} a^2 R \frac{\partial^3}{\partial x^2 \partial s};$$

$$\mathscr{L}_{13} = \mathscr{L}_{31} = \frac{\mu}{R}\frac{\partial}{\partial x} - a^2 R\left(\frac{\partial^3}{\partial x^3} - \frac{1-\mu}{2}\frac{\partial^3}{\partial x \partial s^2}\right);$$

$$\mathscr{L}_{33} = a^2\left(R^2 \nabla^2 \nabla^2 + 2\frac{\partial^2}{\partial s^2} + \frac{1}{R^2}\right) + \frac{1}{R^2},$$

q_x, q_θ, q_r are loading components per unit area of the middle surface, μ is Poisson's ratio and

$$s = R\theta, \qquad a^2 = \frac{h^2}{12R^2}, \qquad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial s^2}. \tag{4.22}$$

Let p_i be the internal and p_e the external pressure on the shell. Then obviously

$$q_x = -\varrho_0 h \frac{\partial^2 u}{\partial t^2}, \qquad q_\theta = -\varrho_0 h \frac{\partial^2 v}{\partial t^2}, \qquad q_r = -\varrho_0 h \frac{\partial^2 w}{\partial t^2} + p_i - p_e. \tag{4.23}$$

It is well-known that for modes of oscillations with a sufficiently large variability characteristic the effect of the tangential components of the inertia forces is negligibly small. It can be shown that the lowest critical flow velocities correspond to motions with a half-wave length λ, which is of the order $\lambda \sim (R h)^{1/2}$. Thus, if the shell is thin ($h/R \ll 1$), the most significant motions are those which are characterized by a large variability characteristic and for which, consequently, tangential inertia forces can be ignored. In this case, introducing the function $\Phi(x, \theta, t)$, which is related to the displacement w in the following way

$$w = R^4 \nabla^2 \nabla^2 \Phi, \tag{4.24}$$

we can reduce the system of equations (4.21) to the single equation

$$\nabla^2 \nabla^2 \nabla^2 \nabla^2 \Phi + \frac{1-\mu^2}{a^2 R^4} \frac{\partial^4 \Phi}{\partial x^4} = \frac{q_r}{R^4 D}. \tag{4.25}$$

Here $D = E h^3/12(1 - \mu^2)$ is the cylindrical stiffness; in writing down Equation (4.25), we omitted terms the effect of which for states with a high variability characteristic is also negligibly small. Taking into account (4.23) and (4.24) we obtain

$$q_r = -\varrho_0 h R^4 \nabla^2 \nabla^2 \frac{\partial^2 \Phi}{\partial t^2} + p_i - p_e. \tag{4.26}$$

Consider now the equations of motion of the fluid. The velocity field for disturbed motion is given by

$$\left.\begin{aligned} v_x &= U_i + \frac{\partial \bar\varphi_i}{\partial x}; \qquad v_\theta = \frac{1}{r}\frac{\partial \bar\varphi_i}{\partial \theta}; \qquad v_r = \frac{\partial \bar\varphi_i}{\partial r};\\ v_x &= U_e + \frac{\partial \bar\varphi_e}{\partial x}; \qquad v_\theta = \frac{1}{r}\frac{\partial \bar\varphi_e}{\partial \theta}; \qquad v_r = \frac{\partial \bar\varphi_e}{\partial r}, \end{aligned}\right\} \tag{4.27}$$

where $\bar{\varphi}_i\,(x, r, \theta, t)$ and $\bar{\varphi}_e(x, r, \theta, t)$ are the potentials of the disturbances for the internal and external regions, respectively. (The suffixes i and e, which indicate whether a quantity belongs to the internal or external region, will be omitted where there is no likelihood of ambiguity.)

If the deformations of the shell are small enough, the disturbed motion caused by them will differ only slightly from the given flow with constant velocities U_i and U_e. With this restriction concerning the smallness of the disturbance, the potentials φ_i and φ_e satisfy the linearized Equation (4.9)[1]:

$$(1 - M^2)\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \\ + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} - \frac{2M}{c_\infty}\frac{\partial^2\varphi}{\partial x\,\partial t} - \frac{1}{c_\infty^2}\frac{\partial^2\varphi}{\partial t^2} = 0. \qquad (4.28)$$

In this equation c_i and c_e, the velocities of sound in the undisturbed flow, are substituted for c_∞; $M_i = U_i/c_i$, $M_e = U_e/c_e$.

Apart from the special boundary conditions, the solutions to Equations (4.21) and (4.28) must satisfy the general boundary conditions on the oscillating wall of the cylinder. The condition of impermeability (4.11) in this case becomes

$$\left.\begin{aligned} \frac{\partial\varphi_i}{\partial r} &= U_i\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} \quad (\text{at } z = R - 0), \\ \frac{\partial\varphi_e}{\partial r} &= U_e\frac{\partial w}{\partial x} + \frac{\partial w}{\partial t} \quad (\text{at } z = R + 0). \end{aligned}\right\} \qquad (4.29)$$

In addition, the potential φ_e, must satisfy the condition of decay of the disturbances at infinity

$$\varphi_e \to 0, \qquad \frac{\partial\varphi_e}{\partial r} \to 0 \quad (\text{as } r \to \infty).$$

If the potential φ_e follows a wave process, these conditions are insufficient for a complete determination of the solution; they must be replaced by the conditions of radiation. These require that the potential φ_e satisfies the condition

$$\varphi_e = O\,(r^{-1/2}),$$

---

[1] Equation (4.9) is written in rectangular Cartesian coordinates; Equation (4.28) is written in cylindrical coordinates $v_x = U$, $v_r = v_\theta = 0$.

i.e. that it describes the propagation of waves the intensity of which decreases with increase in distance from their source as $r^{-1/2}$. Furthermore, and this is the most important feature of the radiation conditions, the potential must describe divergent waves if the shell emits energy and convergent waves if the shell absorbs energy from the flow.

The dynamic condition on the oscillating wall of the cylinder follows from the general condition (4.12):

$$\bar{p}_i = \varrho_i\left(\frac{\partial \bar{\varphi}_i}{\partial t} + U\frac{\partial \bar{\varphi}_i}{\partial x}\right), \qquad \bar{p}_e = -\varrho_e\left(\frac{\partial \bar{\varphi}_e}{\partial t} + U\frac{\partial \varphi_e}{\partial x}\right). \tag{4.30}$$

4.5. Case of an Infinitely Long Shell. Various Types of Flow

We shall now consider an infinitely long shell and seek a solution in the form of waves travelling along the shell. Equations (4.21) can be satisfied by setting

$$\left.\begin{aligned} u &= i\,\xi\, e^{i(\Omega t - kx)}\cos n\,\theta,\\ v &= \eta\, e^{i(\Omega t - kx)}\sin n\,\theta,\\ w &= \zeta\, e^{i(\Omega t - kx)}\cos n\,\theta. \end{aligned}\right\} \tag{4.31}$$

Here ξ, η, ζ are coefficients, in general complex, the wave number $k = \pi/\lambda$, where λ is the half-wave length in the direction of the generators, $n = 0, 1, 2, 3, \ldots$ We have the following expression for the potential $\bar{\varphi}$:

$$\bar{\varphi} = f(r)\, e^{i(\Omega t - kx)}\cos n\,\theta, \tag{4.32}$$

where $f(r)$ is a function to be determined. Substitution into Equation (4.28) gives

$$\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} + \left[\left(\frac{kU - \Omega}{c_\infty}\right)^2 - k^2 - \frac{n^2}{r^2}\right]f = 0. \tag{4.33}$$

We introduce the notation $V = \Omega/k$ for the phase velocity of propagation of an elastic wave in the shell, together with the notation

$$M_1 = \frac{|U - V|}{c_\infty} \tag{4.34}$$

for the Mach number in the relative motion of the gas and the elastic wave. If $M_1 < 0$, then setting $r_0 = \nu\, r$, where

$$\nu = k\sqrt{|M_1^2 - 1|}, \tag{4.35}$$

we obtain, instead of (4.33), the equation

$$\frac{d^2f}{dr_0^2} + \frac{1}{r_0}\frac{df}{dr_0} - \left(1 + \frac{n^2}{r_0^2}\right) f = 0 .$$

This equation can be integrated in Bessel functions of purely imaginary argument of order n:

$$f(r_0) = C_1 I_n(r_0) + C_2 K_n(r_0).$$

If $M_1 > 1$, then in the same way we obtain the equation

$$\frac{d^2f}{dr_0^2} + \frac{1}{r_0}\frac{df}{dr_0} + \left(1 - \frac{n^2}{r_0^2}\right) f = 0,$$

the integral of which can be expressed either in terms of Bessel functions of real argument of order n:

$$f(r_0) = C_1 J_n(r_0) + C_2 N_n(r_0),$$

or in terms of corresponding Hankel functions of the first and second kinds:

$$f(r_0) = C_1 H_n^{(1)}(r_0) + C_2 H_n^{(2)}(r_0).$$

If $M = 1$, degeneration into power functions occurs

$$f(r_0) = C_1 r_0^n + \frac{C_2}{r_0^n} .$$

In general, $f(r_0) = C\, Z_n(r_0)$, where $Z_n(r_0)$ is a cylindrical function of order n. Consequently

$$\bar{\varphi} = C\, Z_n(\nu r)\, e^{i(\Omega t - kx)} \cos n\theta . \tag{4.36}$$

In order to find the constant C, we use Condition (4.29). Substituting (4.31) and (4.30), we find

$$C = \frac{i(\Omega - k\, U)\zeta}{\nu\, Z_n'(\nu R)} ,$$

where

$$Z_n'(\nu R) = \left.\frac{dZ_n(r_0)}{dr_0}\right|_{r_0 = \nu R} .$$

Returning to (4.30), we obtain

$$\bar{\varphi} = \frac{Z_n(\nu R)}{\nu\, Z_n'(\nu R)}\, i(\Omega - k\, U)\zeta\, e^{i(\Omega t - kx)} \cos n\theta .$$

We can now use Formula (4.30) to determine the pressure on the internal and external surfaces of the shell. Setting

$$\alpha_i = -\frac{Z_n(\nu_i R)}{\nu_i R Z_n'(\nu_i R)}; \qquad \alpha_e = -\frac{Z_n(\nu_e R)}{\nu_e R Z_n'(\nu_e R)}, \tag{4.37}$$

where ν_i and ν_e are given by (4.35) by substituting c_i, M_i and c_e, M_e, respectively, we find immediately that

$$\left.\begin{aligned} \bar{p}_i &= \varrho_i R\alpha_i(\Omega - kU_i)^2 \zeta e^{i(\Omega t - kx)} \cos n\theta, \\ \bar{p}_e &= -\varrho_e R\alpha_e(\Omega - kU_e)^2 \zeta e^{i(\Omega t - kx)} \cos n\theta. \end{aligned}\right\} \tag{4.38}$$

Formulae (4.38) can also be put in the form

$$\bar{p} = \pm\varrho_0 R\alpha\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 w(x, \theta, t), \tag{4.39}$$

where the upper sign refers to the internal pressure, the lower sign to the external pressure.

The form of the function $Z_n(r_0)$ can be established by considering the boundary conditions for $\bar{\varphi}$. It is well known that the functions $K_n(r_0)$ and $N_n(r_0)$ are singular at the origin of coordinates. Therefore, for the internal region we must set $C_2 = 0$, or, more exactly,

$$Z_n(\nu r) = \begin{cases} I_n(\nu r) & (M_1 < 1), \\ J_n(\nu r) & (M_1 > 1). \end{cases} \tag{4.40}$$

For the external region the function is chosen from the conditions of damping of the disturbances at infinity or from the conditions of radiation. If $M_1 \leqq 1$, the damping conditions at infinity can be satisfied by setting $C_1 = 0$. The problem becomes more complex if $M_1 > 0$. From the asymptotic expressions for the Bessel functions

$$J_n(r_0) = \sqrt{\left(\frac{2}{\pi r_0}\right)} \cos(r_0 - \delta_n) + O(r_0^{-3/2}),$$

$$N_n(r_0) = \sqrt{\left(\frac{2}{\pi r_0}\right)} \sin(r_0 - \delta_n) + O(r_0^{-3/2})$$

$\delta_n = n\pi/2 - \pi/4)$ it can be seen that as $r_0 \to \infty$ both functions decrease as $r_0^{-1/2}$. Therefore, the conditions of damping of the disturbances at infinity are insufficient in this case to find a unique solution.

The radiation condition can be easily obtained from physical considerations. If $V > U$, then the oscillating shell radiates energy into the surrounding space. Conversely, the motion which occurs $V \leqq U$

(this case is of immediate interest from the point of view of flutter) can be maintained only if energy is absorbed by the shell from the flow. Consider the asymptotic expressions

$$H_n^{(1)}(r_0) \approx \sqrt{\left(\frac{2}{\pi r_0}\right)} e^{i(r_0 - \delta_n)} + O(r_0^{-3/2}),$$

$$H_n^{(2)}(r_0) \approx \sqrt{\left(\frac{2}{\pi r_0}\right)} e^{-i(r_0 - \delta_n)} + O(r_0^{-3/2}).$$

It will be seen that if $V > U$ the radiation conditions are satisfied by the second function only, i.e. that $C_1 = 0$. In fact, with the time factor $e^{i\Omega t}$, the first function gives a solution of the type $f(r_0 + \Omega t)$, i.e. convergent waves, whereas the second function gives a solution of the type $f(r_0 - \Omega t)$. Analogous reasoning for the case of $V < U$ leads to the conclusion that $C_2 = 0$.

Thus, for the external region

$$Z_n(\nu r) = \begin{cases} K_n(\nu r) & (M_1 < 1), \\ H_n^{(2)}(\nu r) & (M_1 > 1,\ V > U), \\ H_n^{(1)}(\nu r) & (M_1 > 1,\ V < U). \end{cases} \tag{4.41}$$

It should be noted that the author's paper[1] considered only one case, namely when the shell radiates energy into space; therefore the latter case ($M_1 > 1$, $V < U$) was not identified.

Using the well-known formulae of differentiation of cylindrical functions, we obtain

$$\frac{1}{\alpha_i} = \begin{cases} \nu R \dfrac{I_{n-1}}{I_n} - n & (M_1 < 1), \\ \nu R \dfrac{J_{n-1}}{J_n} - n & (M_1 > 1). \end{cases} \tag{4.42}$$

Similarly, for the external flow we find that

$$\frac{1}{\alpha_e} = \begin{cases} \nu R \dfrac{K_{n-1}}{K_n} + n & (M_1 < 1), \\ n - \nu R \dfrac{H_{n-1}^{(2)}}{H_n^{(2)}} & (M_1 > 1,\ V > U), \\ n - \nu R \dfrac{H_{n-1}^{(1)}}{H_n^{(1)}} & (M_1 > 1,\ V < U). \end{cases} \tag{4.43}$$

---

[1] See reference on p. 214.

In Formulae (4.42) and (4.43) (and also in those that follow) the cylindrical functions are calculated for $r_0 = \nu R$.

The coefficients α_e for $M_1 > 1$ are complex. We note also that

$$\alpha_e = \begin{cases} -\dfrac{1}{\nu R}\,\dfrac{J_n + i\,N_n}{J_n' + i\,N_n'} & (V > U), \\[2ex] -\dfrac{1}{\nu R}\,\dfrac{J_n - i\,N_n}{J_n' - i\,N_n'} & (V < U). \end{cases}$$

Separating the real and imaginary parts, we can express the coefficient α_e in the form

$$\alpha_e = \alpha_* - i\,\beta\,\operatorname{sgn}(V - U),$$

where

$$\alpha_* = -\frac{1}{\nu R}\,\frac{J_n J_n' + N_n N_n'}{(J_n')^2 + (N_n')^2}, \qquad \beta = \frac{1}{\nu R}\,\frac{J_n N_n' - N_n J_n'}{(J_n')^2 + (N_n')^2}.$$

The coefficient $\alpha^* > 0$ characterizes the reduced mass of the gas and $\beta > 0$ the damping coefficient.[1] Indeed, in the type of motion under consideration, with the time factor $\exp(i\,\Omega\,t)$, we have

$$\alpha_e \frac{\partial^2 w}{\partial t^2} = \alpha_* \frac{\partial^2 w}{\partial t^2} + \beta\,\Omega\,\operatorname{sgn}(V - U)\frac{\partial w}{\partial t},$$

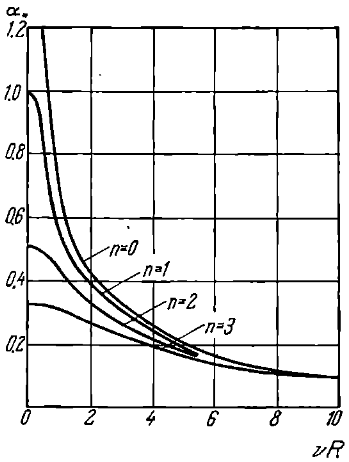


Fig. 102

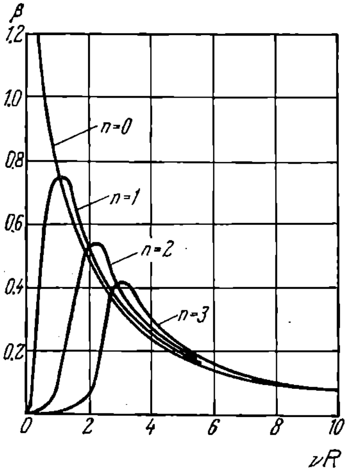


Fig. 103

---

[1] The inequality $\beta > 0$ follows from the fact that the numerator in the formula for β is the Wronskian of the Bessel functions

$$J_n N_n' - N_n J_n' = \frac{2\pi}{\nu R} > 0.$$

where the second term takes into account damping (positive if $V > U$ and negative if $V < U$, which agrees with our principle of energy radiation).

The coefficients $\alpha_i, \alpha_e, \alpha_*$ and β have been evaluated in the paper by the author.[1] Figures 102 and 103 show the graphs of the more important coefficients.

4.6. DETERMINATION OF CRITICAL FLUTTER AND DIVERGENCE VELOCITIES

Returning to the equations of oscillations of a shell (4.21) and substituting (4.23) and (4.39), we obtain

$$\left.\begin{aligned}
&-\frac{\varrho_0 h(1-\mu^2)}{Eh}\frac{\partial^2 u}{\partial t^2} + \mathscr{L}_{11}u + \mathscr{L}_{12}v + \mathscr{L}_{13}w = 0,\\
&-\frac{\varrho_0 h(1-\mu^2)}{Eh}\frac{\partial^2 v}{\partial t^2} + \mathscr{L}_{21}u + \mathscr{L}_{22}v + \mathscr{L}_{23}w = 0,\\
&\frac{\varrho_0 h(1-\mu^2)}{Eh}\left[(1-\chi)\frac{\partial^2 w}{\partial t^2} + 2U\chi\frac{\partial^2 w}{\partial t\,\partial x} + U^2\chi\frac{\partial^2 w}{\partial x^2}\right] +\\
&\qquad\qquad + \mathscr{L}_{31}u + \mathscr{L}_{32}v + \mathscr{L}_{33}w = 0,
\end{aligned}\right\} \quad (4.44)$$

where the notation was introduced

$$\chi = \frac{(\varrho_i\alpha_i + \varrho_e\alpha_e)R}{\varrho_0 h}. \quad (4.45)$$

Substituting Expressions (4.31) into Equations (4.44), we obtain a system of algebraic equations

$$\sum_{k=1}^{3}\left[\frac{E}{\varrho_0 R^2(1-\mu^2)}a_{jk} - \Omega^2\delta_{jk} - \chi(\Omega - kU)^2\delta_{j3}\right]u_k = 0,$$

in which

$$\begin{aligned}
a_{11} &= m^2 + \frac{1-\mu}{2}n^2,\\
a_{22} &= \frac{1-\mu}{2}m^2 + n^2,\\
a_{33} &= a^2[(m^2+n^2)^2 - 2n^2 + 1] + 1,\\
a_{12} &= a_{21} = \frac{1+\mu}{2}mn,\\
a_{13} &= a_{31} = \mu m + a^2 m\left(m^2 - \frac{1-\mu}{2}n^2\right),\\
a_{23} &= a_{32} = n - \frac{3-\mu}{2}a^2 m^2 n,
\end{aligned}$$

---

[1] See reference on p. 214.

and where $m = k R = \pi R/\lambda$, $u_j = (\xi, \eta, \zeta)$, $\delta_{j3} = (0, 0, 1)$. Consequently, in this case Equation (4.20) assumes the form

$$\left| \frac{E\, a_{jk}}{\varrho_0 R^2(1 - \mu^2)} - \Omega^2 \delta_{jk} - \chi(\Omega - k\, U)^2 \delta_{j3} \right| = 0. \tag{4.46}$$

If $U = 0$, Equation (4.46) enables us to find the natural frequency of oscillation in a fixed acoustic medium. With fixed values of m and n the equation

$$\left| \frac{E}{\varrho_0 R^2(1 - \mu^2)}\, a_{jk} - \Omega^2 \delta_{jk} \right| = 0$$

has three roots $\Omega_1, \Omega_2, \Omega_3$ corresponding to the natural frequencies *in vacuo* with given wave numbers. Suppose that the matrix of the coefficients ξ, η and ζ for natural oscillations *in vacuo* is of the form

$$\begin{Vmatrix} \xi_1 & \eta_1 & 1 \\ \xi_2 & \eta_2 & 1 \\ \xi_3 & \eta_3 & 1 \end{Vmatrix}.$$

Then the matrix

$$v_{jk} = \begin{Vmatrix} \xi_1 \sqrt{\gamma_1} & \eta_1 \sqrt{\gamma_1} & \sqrt{\gamma_1} \\ \xi_2 \sqrt{\gamma_2} & \eta_2 \sqrt{\gamma_2} & \sqrt{\gamma_2} \\ \xi_3 \sqrt{\gamma_3} & \eta_3 \sqrt{\gamma_3} & \sqrt{\gamma_3} \end{Vmatrix}.$$

where

$$\gamma_\alpha = \frac{1}{\xi_\alpha^2 + \eta_\alpha^2 + 1},$$

will be an orthonormalized matrix, and will reduce the matrix a_{jk} to the diagonal form

$$v_{\alpha j}\, a_{jk}\, v_{\beta k} = \Omega_\alpha^2 \delta_{\alpha\beta}.$$

Noting also that

$$v_{\alpha j}\, \delta_{j3}\, v_{\beta 3} = v_{\alpha 3}\, v_{\beta 3} = \sqrt{(\gamma_\alpha \gamma_\beta)},$$

we obtain in place of (4.46) the equation

$$|(\Omega_\alpha^2 - \Omega^2)\, \delta_{\alpha\beta} - \chi(\Omega - k\, U)^2 \sqrt{(\gamma_\alpha \gamma_\beta)}| = 0.$$

This equation can also be written in the form

$$\sum_{\alpha=1}^{3} \frac{\gamma_\alpha}{\Omega_\alpha^2 - \Omega^2} = \frac{1}{\chi(\Omega - k\, U)^2} \qquad (n \neq 0). \tag{4.47}$$

If $n = 0$, the number of terms on the left-hand side reduces to two.

If the wave numbers n are large enough, one of the natural frequencies (namely the one which corresponds to primarily bending oscillations) is considerably less than the other two. Suppose that $\Omega_1^2 \ll \Omega_2^2$, $\Omega_1^2 \ll \Omega_3^2$, $\gamma_3 \approx 1$. Then Equation (4.47) can be written in the form

$$\Omega_1^2 - \Omega^2 - \chi(\Omega - k\,U)^2 = 0. \tag{4.48}$$

We can derive Equation (4.48) by starting with Equation (4.30) for a shallow shell and substituting in its right-hand side not (4.26) but the expression

$$q_r = \varrho_0\, h\, R^4 \left[\frac{\partial^2}{\partial t^2} + \chi\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\right] \nabla^2\,\nabla^2\,\Phi$$

and letting

$$\Phi = \Phi_0\, e^{i(\Omega t - k x)} \cos n\,\theta$$

Substitution yields an equation of the same type as (4.48):

$$(m^2 + n^2)^4 + \frac{1-\mu^2}{a^2} m^4 - \frac{\varrho_0\, h(m^2 + n^2)^2\, R^4}{D} \left[\Omega^2 + \chi(\Omega - k\,U)^2\right] = 0, \tag{4.49}$$

where, as before $m = k\,R$. The natural frequency of primarily bending oscillations in vacuo is given by the formula

$$\Omega_1 = \frac{1}{R^2}\sqrt{\left(\frac{D}{\varrho_0\, h}\right)} F(m, n)\cdot \tag{4.50}$$

Here

$$F(m, n) = \sqrt{\left[(m^2 + n^2)^2 + \frac{1-\mu^2}{a^2}\,\frac{m^4}{(m^2+n^2)^2}\right]}. \tag{4.51}$$

In order to find the critical velocity of divergence we put $\Omega = 0$ in Equation (4.47). In this way we obtain the formula

$$U_* = \left(k\,\chi \sum_{\alpha=1}^{3} \frac{\gamma_\alpha}{\Omega_\alpha^2}\right)^{-1/2}.$$

By starting from the simplified equation, we find that

$$U_* \approx \frac{V_1}{\sqrt{\chi}}. \tag{4.52}$$

Here V_1 is the phase velocity of propagation of primarily bending waves for natural oscillations *in vacuo:*

$$\left.\begin{aligned} V_1 &= \frac{\Omega_1}{k} = \frac{1}{R}\sqrt{\left(\frac{D}{\varrho_0\, h}\right)} F_1(m, n),\\ F_1(m, n) &= \sqrt{\left[\frac{(m^2+n^2)^2}{m^2} + \frac{1-\mu^2}{a^2}\,\frac{m^2}{(m^2+n^2)^2}\right]}. \end{aligned}\right\} \tag{4.53}$$

Since the parameter χ also depends on the velocity U, the calculations need to be carried out by a method of successive approximations. Alternatively, a graphical method may be used.[1]

Obviously, a critical divergence velocity exists only when the parameter χ is real. In the case of an external supersonic flow ($M = M_1 > 1$) Formulae (4.43) show that it is impossible for divergence to occur. If we confine our attention to forms of loss of stability which are of an essentially bending nature, which in any case are of most interest, we can impose a further condition that $\chi > 0$.

Calculations show that the minimum velocity of propagation of bending waves *in vacuo* for a shell of thickness $h \sim 0.01\ R$ is of the order of the speed of sound in a gas and can even exceed this speed. Indeed, the function $F_1(m, n)$ has a minimum when

$$\frac{m^2+n^2}{m} = \frac{\sqrt[4]{(1-\mu^2)}}{\sqrt{a}} \approx \frac{1}{a^{1/2}} \tag{4.54}$$

which is given by (Fig. 104)

$$\min F_1 \approx \left(\frac{2}{a}\right)^{1/2}.$$

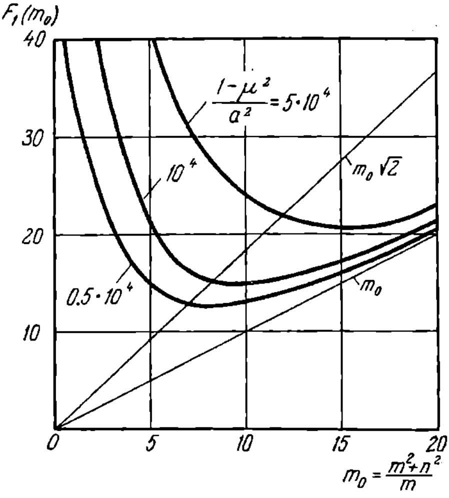


Fig. 104

---

[1] Bolotin, V. V. Some new problems in shell dynamics. *Sb.* "*Raschety na prochnost'*", No. 4, Mashgiz, 1959.

Suppose that $m^2 \gg n^2$. Expression (4.54) then becomes $m \approx 1/a^{1/2}$, which gives an idea of the half-wave length corresponding to the minimum phase velocity:

$$\lambda \approx 1.68 (R\,h)^{1/2}. \tag{4.55}$$

The actual velocity is given approximately by

$$V_1 \approx 1.07 \left(\frac{E}{\varrho_0}\,\frac{h}{R}\right)^{1/2}, \tag{4.56}$$

and the corresponding frequency Ω_1 by

$$\Omega_1 = k\,V_1 \approx 1.99 \frac{1}{R} \left(\frac{E}{\varrho_0}\right)^{1/2}. \tag{4.57}$$

Suppose, for example, that $E = 2 \times 10^6\,\mathrm{kg/cm^2}$, $\varrho_0 = 8 \times 10^{-6}\,\mathrm{kg\,sec^2/cm^4}$ (steel), $h = 0.01\,R$. Then from Formula (4.56) $V_1 = 5.4 \times 10^4$ cm/sec, which is close to the speed of sound in air ($c_\infty = 3.3 \times 10^4$ cm/sec).

Let us now find the critical flutter velocity for the case when $M_1 > 1$. In addition to aerodynamic damping we shall also take into account structural damping with a coefficient ε. Instead of Equation (4.48) we obtain

$$\Omega_1^2 + i\,\varepsilon\,\Omega^2 - \Omega^2 - \chi(\Omega - k\,U)^2 = 0. \tag{4.58}$$

If $U = 0$, all values of the frequency Ω lie in the upper half-plane of the complex variable, which corresponds to damping of the disturbances of the type (4.31). The flutter velocity is determined here as the minimum velocity at which travelling waves of progressing amplitude occur. Amongst the frequencies Ω there are those which have a negative imaginary part, i.e. the corresponding characteristic exponents $s = i\,\Omega$ have positive real parts.

The parameter χ is a complex transcendental function of the velocity U and frequency Ω. Nevertheless, the critical velocity can be found if the well-known Cauchy–Nyquist criteria etc. are applied to Equation (4.58). With sufficiently large values of M_1 the expression for χ can be replaced by a more simple asymptotic expression, and it is then possible to apply algebraic stability criteria and thus obtain an estimate of the critical velocity in explicit form.

Consider the case when

$$M_1^2 \gg 1, \qquad \nu\,R = k\,R\sqrt{(M_1^2 - 1)} \gg n. \qquad (4.59$$

The first condition requires the numbers M_1 in the relative motion of the gas and the elastic wave to be sufficiently large. The second requires the variability parameter across the flow to be not too large.

We shall start from the last of Formulae (4.43):

$$\alpha_e = \frac{1}{n - \nu R \dfrac{H^{(1)}_{n-1}(\nu R)}{H^{(1)}_{n}(\nu R)}} \tag{4.60}$$

(we are not interested in the case when $V > U$, since the shell does not then absorb energy, but radiates it into the surrounding medium). Taking into account the asymptotic formulae (4.43), we find that

$$\frac{H^{(1)}_{n-1}(\nu R)}{H^{(1)}_{n}(\nu R)} = \frac{e^{-\frac{i(n-1)\pi}{2}}}{e^{-\frac{in\pi}{2}}} + O\left(\nu R^{-\frac{3}{2}}\right) = i + O\left(\nu R^{-\frac{3}{2}}\right).$$

Thus, from Formula (4.60) we obtain the asymptotic expression

$$\alpha_e \approx \frac{1}{n - i k R \sqrt{(M_1^2 - 1)}}$$

or, taking into account Inequalities (4.59) and Formulae (4.34) and (4.45),

$$\alpha_e \approx \frac{i}{k R M_1}, \qquad \chi = \frac{\varrho\, c_\infty}{\varrho_0\, h}\, \frac{i}{k U - \Omega}.$$

Equation (4.58) becomes

$$\Omega^2 - i\,\Omega(\varepsilon + \gamma) - \Omega_1^2 + i \gamma k U = 0, \tag{4.61}$$

where

$$\gamma = \frac{\varrho\, c_\infty}{\varrho_0\, h}. \tag{4.62}$$

Equation (4.61) is an algebraic equation in Ω with complex coefficients. The conditions that this equation has no roots with negative imaginary parts can be expressed in a form analogous to the well-known Routh-Hurwitz criteria.[1]

---

[1] Chebotarev, N. G. and Meiman, N. S. *The Routh-Hurwitz Problem for Polynomials and Complete Functions* (Problema Rausa-Hurwitza dlia polinomov i tselykh funktsii). Izd-vo. Akad. Nauk USSR, 1949.

All the roots of the equation

$$F(\Omega) = a_0\Omega^n + a_1\Omega^{n-1} + \cdots + a_n + \\ + i(b_0\Omega^n + b_1\Omega^{n-1} + \cdots + b_n) = 0,$$

where $a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n$ are real numbers, have positive imaginary parts if all the even-order principal minors of the matrix

$$H = \left\| \begin{array}{cccccc} a_0 & a_1 \ldots a_n & & & & \\ b_0 & b_1 \ldots b_n & & & & \\ & a_0 & a_1 \ldots a_n & & & \\ & b_0 & b_1 \ldots b_n & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & a_0 & a_1 \ldots a_n \\ & & & & b_0 & b_1 \ldots b_n \end{array} \right\|$$

are positive. The matrix H in the present case is

$$H = \left\| \begin{array}{cccc} 1 & 0 & -\Omega_1^2 & 0 \\ 0 & -(\varepsilon+\gamma) & \gamma k U & 0 \\ 0 & 1 & 0 & -\Omega_1^2 \\ 0 & 0 & -(\varepsilon+\gamma) & \gamma k U \end{array} \right\| .$$

The stability conditions can be written in the form

$$\varepsilon + \gamma > 0, \qquad (2\varepsilon + \gamma)^2\Omega^2 - (\gamma k U)^2 > 0.$$

The first condition, which requires the damping (internal and aerodynamic) to be positive, is satisfied in every case. The second gives

$$\frac{\gamma U}{\varepsilon + \gamma} < V_1(m, n), \tag{4.63}$$

where V_1 is the phase velocity of propagation of elastic waves in a shell, defined by Formula (4.53). Thus, taking into account (4.62), we find that the critical velocity is given by[1]

$$U_* = V_1(m, n)\left(1 + \frac{\varrho_0}{\varrho}\,\frac{\varepsilon h}{c_\infty}\right). \tag{4.64}$$

---

[1] This formula was derived in the application of the "law of plane sections" to aerodynamic forces in the paper: Bolotin, V. V. Some new problems in shell dynamics. *Sb. "Raschety na prochnost"*, No. 4, Mashgiz, 1959. The limits of its applicability (4.59) are stated for the first time.

It should be noted that in any case the wave velocity on the boundary of the region of stability coincides with the velocity V_1. For, by substituting in Equation (4.61) according to (4.63),

$$\varepsilon + \gamma = \frac{\gamma k U}{\Omega^2},$$

we immediately find that $\Omega = \Omega_1$, and consequently that $V = V_1$.

In this problem the effect of structural damping is considerable. In fact, the second term in brackets in Formula (4.64) can be expressed in the form

$$G = \frac{\varrho_0}{\varrho} \frac{\varepsilon h}{a} = \frac{\psi}{2\pi} \frac{\varrho_0}{\varrho} \frac{\Omega_1 h}{c_\infty},$$

where ψ is the relative energy dissipation resulting from structural damping. Substituting the frequency Ω_1 given by Formula (4.57) and the velocity of sound in gas

$$c_\infty = \sqrt{\frac{\varkappa p_\infty}{\varrho}} \tag{4.65}$$

(p_∞ is the pressure, $\varkappa$ is the polytropy index), we obtain

$$G \approx \frac{\psi}{2\pi} \frac{1.99\, h}{R} \left(\frac{E}{\varkappa p_\infty} \frac{\varrho_0}{\varrho} \right)^{1/2}. \tag{4.66}$$

Suppose that $E = 2 \times 10^6$ kg/cm^2, $\varrho_0 = 8 \times 10^{-6}$ kg sec^2/cm^4 (steel), $p_\infty = 1$ kg/cm^2, $\varrho = 1.3 \times 10^{-9}$ kg sec^2/cm^4, $\varkappa = 1.4$ (air under normal conditions) $h = 0.01\, R$, $\psi = 0.02$ (comparatively little structural damping). Then from Formula (4.66) we find that $G = 2.99$.

From this example we see that the critical velocity of the gas must usually be several times larger than the minimum velocity of bending-wave propagation in the shell, in order for excitation of wave motion to be possible. The Mach number M_1, in relative motion for shells which are not too thin, would be very much larger than unity, and the first of Conditions (4.59) would thus be satisfied. In fact

$$M_1 = \frac{U_* - V_1}{c_\infty} = \frac{V_1}{c_\infty} G,$$

and with Formulae (4.56) and (4.65) we have

$$\frac{V_1}{c_\infty} \approx 1.07 \left(\frac{E}{\varkappa p_\infty} \frac{\varrho}{\varrho_0} \frac{h}{R} \right)^{1/2}.$$

Using the data of the previous example we find that $V_1/c_\infty \approx 1.64$ and $M_{1*} \approx 4.89$.

It is important to note that the results presented above refer to the case of an infinitely long shell. If the shell is of finite length, then as a result of reflection of the elastic waves at its ends, the picture becomes much more complicated. For this reason the formulae given here for the critical velocities might not, of course, coincide with the corresponding formulae for finite, although extremely long, shells.[1]

4.7. STABILITY OF ELASTIC PLATES IN POTENTIAL FLOW

Suppose that an elastic plate, one side of which is exposed to a flow of gas, performs small oscillations. Let $w(x, y, t)$ be the deflection, D the cylindrical stiffness, N_x and N_y the tensile forces in the middle surface, ϱ_0 the density of the material, ε the damping coefficient, $\bar{p}(x, y, t)$ the component of the aerodynamic pressure caused by the deviation of the plate from its undisturbed state. The equation of small oscillations of the plate is

$$D \nabla^2 \nabla^2 w - \left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2}\right) + \varrho_0 h \frac{\partial^2 w}{\partial t^2} + \varrho_0 h \varepsilon \frac{\partial w}{\partial t} + \bar{p}(x, y, t) = 0. \tag{4.67}$$

Following mainly the procedure adopted by Nelson and Cunningham[2], we consider the stability of a flat plate of infinite length in a direction perpendicular to the flow, mounted on an absolutely rigid diaphragm and undergoing cylindrical deflection (Fig. 105). We assume a plane, potential flow. The potential of the disturbances φ must satisfy Equation (4.9)

$$\frac{d^2\bar{\varphi}}{\partial x^2}(1 - M^2) + \frac{\partial^2 \bar{\varphi}}{\partial z^2} - \frac{2M}{c_\infty} \frac{\partial^2 \bar{\varphi}}{\partial x \, \partial t} - \frac{1}{c_\infty^2} \frac{\partial^2 \bar{\varphi}}{\partial t^2} = 0 \tag{4.68}$$

and the condition

$$\frac{\partial \bar{\varphi}}{\partial z} = v(x, t) \quad \text{at } z = 0.$$

---

[1] A discussion of this problem is given in the paper: MILES, J. W. On supersonic flutter of long panels, *J. Aerospace Sci.*, 27, No. 6 (1960).

[2] NELSON, H. C. and CUNNINGHAM, H. J. Theoretical investigation of flutter of two-dimensional flat panels with one surface exposed to supersonic potential flow. *NACA Rep.* No. 1280)1956).

Suppose that the plate performs harmonic oscillations with frequency Ω:

$$\left.\begin{aligned} w(x, t) &= W(x)\, e^{i\Omega t}, \\ v(x, t) &= V(x)\, e^{i\Omega t} = \left(i\Omega\, W + U\frac{\partial W}{\partial x}\right) e^{i\Omega t}. \end{aligned}\right\} \tag{4.69}$$

Then for the potential $\bar{\varphi}(x, z, t)$ we can also take the expression

$$\bar{\varphi}(x, z, t) = f(x, z)\, e^{i\Omega t},$$

and Equation (4.68) becomes

$$\frac{\partial^2 f}{\partial x^2}(1 - M^2) + \frac{\partial^2 f}{\partial z^2} - \frac{2iM\Omega}{c_\infty}\frac{\partial f}{\partial x} + \frac{\Omega^2}{c_\infty^2} f = 0. \tag{4.70}$$

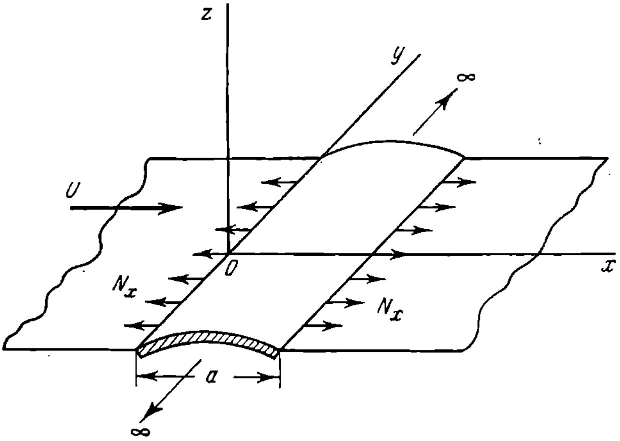


Fig. 105

If $x < 0$ we can set $f = \partial f/\partial z = 0$, which makes the Laplace transform very convenient. Setting

$$f^*(q, z) = \int_0^\infty e^{-qx} f(x, z)\, dx,$$

we reduce Equation (4.70) to the form

$$\frac{\partial^2 f^*}{\partial z^2} = \mu^2 f^*, \qquad \mu = \sqrt{\left(q^2(M^2 - 1) + 2q\frac{iM\Omega}{c_\infty} - \frac{\Omega^2}{c_\infty^2}\right)}.$$

Its solution, which is damped as $z \to \infty$ and which satisfies the transformed boundary condition, is of the form

$$f^*(q, z) = -\frac{V^*(q)}{\mu} e^{-\mu z},$$

where $V^*(q)$ is the Laplace transform for the function $V(x)$. The right-hand side is the product of two functions of q; therefore for inversion we can apply the theorem of convolution. After carrying out the computations we obtain

$$f(x, 0) = -\frac{1}{\sqrt{(M^2 - 1)}} \int\limits_0^x V(\xi) \times$$

$$\times \exp\left[-\frac{iM\Omega(x-\xi)}{c_\infty(M^2-1)}\right] J_0\left[\frac{\Omega(x-\xi)}{c_\infty(M^2-1)}\right] d\xi,$$

where J_0 is a zero-order Bessel function. Thus from Formula (4.12) we easily find the pressure on the oscillating plate, namely,

$$\bar{p} = -\varrho\, e^{i\Omega t}\left[i\Omega f(x, 0) + U\frac{\partial f(x, 0)}{\partial x}\right].$$

We see that the analysis of the stability of the plane form of a panel in a potential supersonic flow reduces to a study of the integro-differential equation

$$D\frac{d^4 W}{dx^4} - N_x \frac{d^2 W}{dx^2} - \varrho_0 h(\Omega^2 + i\varepsilon\Omega) W +$$

$$+ \frac{\varrho}{\sqrt{(M^2-1)}}\left(i\Omega + U\frac{d}{dx}\right)\int\limits_0^x \left[i\Omega W(\xi) + U\frac{dW(\xi)}{d\xi}\right] \times$$

$$\times \exp\left[-\frac{iM\Omega(x-\xi)}{c_\infty(M^2-1)}\right] J_0\left[\frac{\Omega(x-\xi)}{c_\infty(M^2-1)}\right] d\xi \tag{4.71}$$

with the corresponding boundary conditions for $W(x)$. For example, in the case of supported edges we have

$$W(0) = W(a) = \frac{d^2 W(0)}{dx^2} = \frac{d^2 W(a)}{dx^2} = 0. \tag{4.72}$$

The frequencies Ω for which the homogeneous boundary-value problem defined by Equation (4.71) and the boundary Conditions (4.72) has a solution other than the trivial are, in general, complex. On the boundary of the region of stability, however, they assume real values. On this basis we can find the critical flutter velocity.

We express the solution to Equation (4.71) in the form of an expansion in functions satisfying Conditions (4.72), as follows

$$W(x) = \sum_{k=1}^{\infty} f_k \sin \frac{k \pi x}{a}. \tag{4.73}$$

Substituting (4.73) into Equation (4.71) and adopting Galerkin's method, we obtain a system of algebraic equations

$$(\omega_j^2 - \omega^2 + i g \omega) f_j + \sum_{k=1}^{\infty} b_{jk}(\omega, M) = 0 \quad (j = 1, 2, \ldots). \tag{4.74}$$

Here

$$\omega_j = \frac{\Omega_j}{\Omega_0}, \quad \omega = \frac{\Omega}{\Omega_0}, \quad g = \frac{\varepsilon}{\Omega_0},$$

Ω_j are the natural frequencies of the plate *in vacuo*

$$\Omega_j = \frac{j^2}{a^2} \sqrt{\left[\frac{D}{\varrho_0 h}\left(1 + \frac{N_x a^2}{j^2 \pi^2 D}\right)\right]},$$

$b_{jk}(\omega, M)$ are non-dimensional coefficients given by the formula

$$b_{jk} = \frac{2}{\varrho_0 h a \Omega_0^2} \int_0^a \bar{p}_k(x, \omega, M) \sin \frac{j \pi x}{a} dx.$$

In this formula, instead of $\bar{p}_k$, we substitute the last term of Equation (4.71) for the case of

$$W(x) = \sin \frac{k \pi x}{l}.$$

We obtain the equation for finding Ω by equating to zero the determinant of the infinite system (4.74)[1]:

$$|(\omega_j^2 - \omega^2 + i g \omega) \delta_{mn} + b_{jk}(\omega, M)| = 0. \tag{4.76}$$

---

[1] Miles has shown (MILES, J. On a reciprocity condition for supersonic flutter, *J. Aeronaut. Sci.*, **24**, No. 2 (1957)) that in the case of a quasi-stationary approximation to the aerodynamic forces the coefficients b_{jk} satisfy the "reciprocity conditions"

$$b_{jk} = (-1)^{j+k} b_{kj}, \tag{4.75}$$

which are typical conditions also for other non-conservative problems of elastic stability. An analogous result was obtained by Hedgepeth (Flutter of rectangular simply supported panels at high supersonic velocities. *J. Aeronaut. Sci.*, **24**, No. 8 (1957)) also based on quasi-stationary (although three-dimensional) concepts.

The coefficients b_{jk} cannot be expressed explicitly in terms of ω and M; consequently, the determination of the critical velocity involves extremely laborious calculations. Nelson and Cunningham carried out these calculations within the interval $M = 1.3$ to $M = 2.0$ for the case of a simply supported plate and also for a plate with fully fixed ends. In the latter case they used "beam" functions as approximations, which meant that the infinite determinant (4.76) could be replaced by a second- or fourth-order determinant. Figure 106 shows the results of their calculations in the form of a graph of the ratio h/a required to avoid flutter for various values of M. The graph shown is for a panel of aluminium alloy at an altitude of 25,000 ft (7.62 km). It will be noted that the smallest thickness corresponds to values of M close to $\sqrt{2}$. Experimental points are also shown for flat panels (indicated by circles) and for convex panels (indicated by squares).

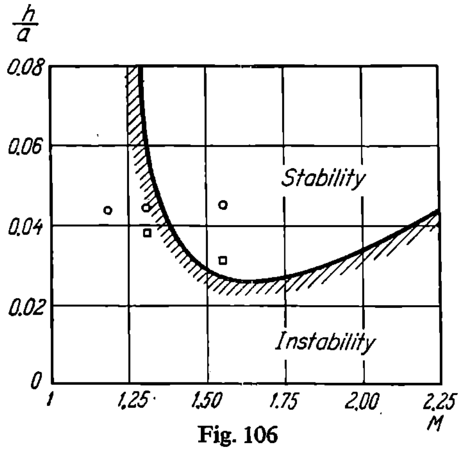


Fig. 106

Exact solutions can be obtained to Equation (4.71) with the aid of Laplace transforms. Goland and Luke[1] have investigated the case of a membrane ($D = 0$). They considered only the limiting case as $M \to \infty$ and showed that at sufficiently high supersonic velocities the plane form of the membrane is stable. The problem of the stability of a plate presents much greater difficulties. Nelson and Cunningham[2] derived an extremely cumbersome equation for finding the critical flutter velocity but did not carry out any calculations.

---

[1] Goland, M. and Luke, Y. L. An exact solution for two-dimensional linear flutter at supersonic speeds. *J. Aeronaut. Sci.*, **21**, No. 2 (1954).

[2] See reference on p. 231.

4.8. Determination of Aerodynamic Forces in the Case of High Supersonic Velocities. Law of Plane Sections

The preceding sections have shown that the solution of problems on the stability of elastic bodies in a flow of gas is made difficult by the complicated expressions for the non-stationary aerodynamic forces in a disturbed flow. Fortunately, at sufficiently high supersonic velocities, which nowadays present most interest, the aerodynamic side of the problem is considerably simplified (provided, of course, that we do not take into account factors associated with viscosity, dissociation, phase changes on the boundary between the body and the flow, etc.). In the present section we shall acquaint ourselves with a few simplified methods.

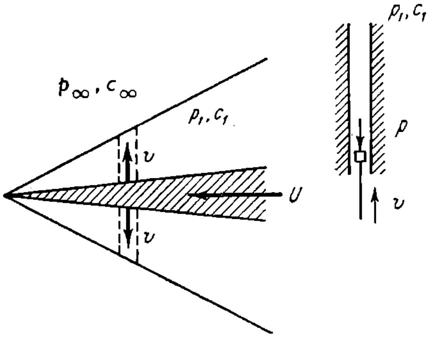


Fig. 107

Consider first the steady motion of a thin profile at a supersonic velocity U (Fig. 107). In contrast to the subsonic case, in which the velocity field at every point depends on the normal component of velocity v at all points on the surface of the body, the disturbances here are transmitted only in a downstream direction. As the velocity U increases, the disturbances assume a more local character, and in the limiting case of very high supersonic velocities each particle of gas moves only in a direction practically perpendicular to the velocity U. It is as if the profile cuts through the gas, particles of which move in narrow bands bounded by extremely close vertical planes. The greater the velocity U, the more exactly does this "*law of plane sections*" hold.

Accepting this description of the phenomenon, we conclude that the pressure p on the surface of the body can be calculated in the same way as the pressure on a piston moving in a one-dimensional duct:[1]

$$p = p_\infty \left(1 + \frac{\varkappa - 1}{2} \frac{v}{c_\infty}\right)^{\frac{2\varkappa}{\varkappa - 1}}. \tag{4.77}$$

here $\varkappa$ is the polytropy index, p_∞ and c_∞ are the pressure and velocity of sound in the undisturbed gas. If the pressure is required on the part of the profile behind the shock wave, p_∞ and c_∞ are, in general, replaced by the parameters p_1 and c_1 for the gas behind the shock wave.

Formula (4.77) and modifications of this formula have been derived by a number of authors from different considerations[2] and have proved to be a very convenient means of investigating problems of aeroelasticity. They can also be regarded as a corollary of the laws of similitude at high supersonic velocities.[3] Setting aside any discussion of the limits of applicability of formulae of the type (4.77), we shall now consider a few particular cases.

Expanding the binomial in (4.77) in a series, we obtain the Formula

$$p = p_\infty \left[1 + \frac{\varkappa v}{c_\infty} + \frac{\varkappa(\varkappa + 1)}{4}\left(\frac{v}{c_\infty}\right)^2 + \frac{\varkappa(\varkappa + 1)}{12}\left(\frac{v}{c_\infty}\right)^3 + \cdots\right]. \tag{4.78}$$

The linear approximation corresponds to the well-known formula of Ackeret[4] for the stationary pressure on a thin profile in a supersonic flow:

$$p - p_\infty = \frac{\varrho_\infty U^2 \eta}{\sqrt{(M^2 - 1)}}$$

(η is the angle of inclination of the profile). If we interpret Ackeret's formula in its wider sense, treating η as a characteristic of the local

---

[1] Hence the term "piston theory", which has now been extended to various approximate methods relating pressure to a local inclination of the flow.

[2] Il'yushin, A. A. The law of plane sections at high supersonic velocities. *PMM*, **20**, No. 6 (1956); Lighthill, M. J. Oscillating airfoil at high Mach number. *J. Aeronaut. Sci.*, **20**, No. 6 (1953); Ashley, H. and Zartarian, C. Piston theory—a new aerodynamic tool for the aeroelastician. *J. Aeronaut. Sci.*, **23**, No. 6 (1956).

[3] Hayes, W. D. On hypersonic similitude. *Quart. Appl. Math.*, **5**, No. 1 (1947); Bam-Zelikovich, G. M., Bunimovich, A. I. and Mikhailova, M. P. Thin bodies travelling at high supersonic velocities. *Izv. Akad. Nauk USSR, OTN*, "*Mekhanika i mashinostroenie*", No. 1 (1960); Chernyi, G. G. *Flow of Gas at a High Supersonic Velocity* (Techenie gaza s bol'shoi sverkhzvukovoi skorost'iu). Fizmatgiz, 1959.

[4] See, for example, Ferri, A. *Elements of the Aerodynamics of Supersonic Flows*. Macmillan, New York 1949.

"inclination" angle of attack of the flow which can vary along the length of the profile, we obtain the formula

$$p - p_\infty = \frac{\varrho_\infty U v}{\sqrt{(M^2 - 1)}}, \tag{4.79}$$

which, if $M^2 \gg 1$, becomes a linear approximation of Formula (4.78). Similarly, the quadratic approximation corresponds to the well-known formula of Busemann when $M^2 \gg 1$.

Let us examine the third-order approximation in a little more detail. It is necessary to do so, since the change in entropy in passing through the shock wave starts with third-order terms. It is therefore in these terms that the difference will exist between Formula (4.77) and the formula

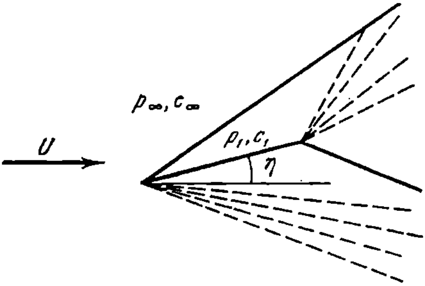


Fig. 108

$$p = p_1\left(1 + \frac{\varkappa - 1}{2}\,\frac{v - v_0}{c_1}\right)^{\frac{\varkappa}{\varkappa - 1}}, \tag{4.80}$$

proposed by Il'iushin[1] (v_0 is the inclination angle of attack of the flow at the leading edge of the profile).

To compare Formulae (4.77) and (4.80) it is convenient to consider the case of a flat plate slightly inclined to the flow (Fig. 108). If $v = v_0$, then the pressure at every point on the plate behind the shock wave will be constant, and from Formula (4.80) equal to p_1, where p_1 is given by

$$p_1 = p_\infty\left\{1 + \frac{\varkappa(\varkappa + 1)}{4}\left(\frac{v}{c_\infty}\right)^2 + \varkappa\frac{v}{c_\infty}\sqrt{\left(1 + \left[\frac{(\varkappa + 1)\,v}{4c_\infty}\right]^2\right)}\right\}.$$

---

[1] See reference on p. 237.

Expanding the right-hand side in a series, we find that

$$p_1 = p_\infty \left[1 + \frac{\varkappa v}{c_\infty} + \frac{\varkappa(\varkappa + 1)}{4} \left(\frac{v}{c_\infty} \right)^2 + \frac{\varkappa(\varkappa + 1)^2}{32} \left(\frac{v}{c_\infty} \right)^3 + \cdots \right].$$

The first three terms in the square brackets coincide with (4.78); the difference between the last terms can be estimated by comparing $(5 - 3\varkappa)/8$ with unity. If $\varkappa = 1.4$, the difference between the cubic terms is 10%. Bearing in mind that the contribution of these terms to the total pressure is very small, we conclude that it is acceptable

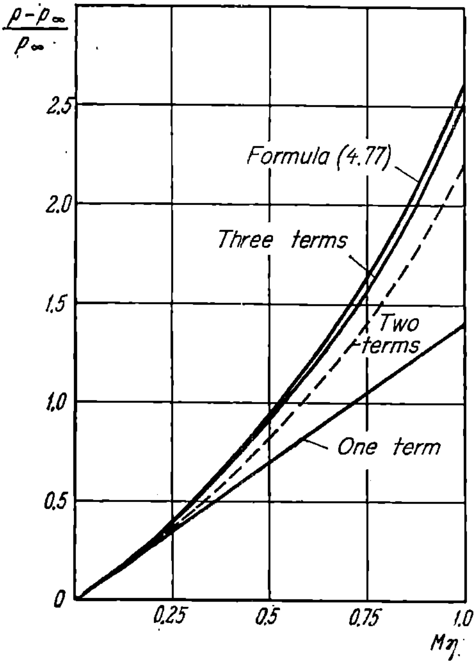


Fig. 109

in a large number of cases to use the comparatively simple Formula (4.77), instead of the more exact Formula (4.80).

Figure 109 shows the variation of the excess pressure evaluated on the basis of Formula (4.78) for various numbers of terms of the series ($\varkappa = 1.4$). We have used the notation

$$\eta = \frac{v}{U}. \tag{4.81}$$

The effect of terms of order higher than the third within the range $0 \leqq M\eta \leqq 1.0$ is seen to be negligible.

The questions of the limits of applicability of the "law of plane sections" has been considered by a number of authors. Apart from the requirement that $M^2 \gg 1$, the condition that the disturbances η are small must also be satisfied. This condition is formulated differently, e.g. $M\eta < 1$, $\eta^2 \ll 1$ etc. According to Il'iushin the error in the approximate formulae is given by a quantity of the order of

$$\frac{1}{2}\left(\eta^2 + \frac{1}{M^2}\right).$$

In addition to the limitations expressed in terms of M and η, further limitations must be imposed which take into account the non-stationary and the three-dimensional nature of the flow. That these limitations exist can be seen by comparing the formulae for the pressure derived from (4.77) and those by a limiting process from the exact relations.

Consider a linearized flow of gas over a shell through which elastic waves are propagated. In this case

$$v = \frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}, \tag{4.82}$$

and consequently, from Formula (4.78), the excess pressure is given by

$$\bar{p} = p - p_\infty = \frac{\varkappa p_\infty}{c_\infty}\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right). \tag{4.83}$$

Substituting the Expression (4.31) for w,

$$w = \zeta\, e^{i(\Omega t - kx)} \cos n\theta, \tag{4.84}$$

we find that

$$\bar{p} = \varrho\, c_\infty\, i(\Omega - k\,U)\,\zeta\, e^{i(\Omega t - kx)} \cos n\theta. \tag{4.85}$$

Taking now the exact Formula (4.39),

$$\bar{p} = \varrho\, R\alpha\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 w.$$

and substituting Expression (4.84), we find that

$$\bar{p} = -\varrho\, R\alpha(\Omega - k\,U)^2 \zeta\, e^{i(\Omega t - kx)} \cos n\theta. \tag{4.86}$$

For the case under discussion

$$\alpha = \frac{1}{n - \nu R \dfrac{H^{(1)}_{n-1}(\nu R)}{H^{(1)}_{n}(\nu R)}},$$

where

$$\nu = k \sqrt{(M_1^2 - 1)}, \qquad M_1 = \frac{U - V}{c_\infty}, \qquad V = \frac{\Omega}{k}.$$

In order to pass from Formula (4.86) to Formula (4.85) it is not sufficient simply to set $M^2 \gg 1$. It is essential that $M_1^2 \gg 1, k R M_1 \gg n$. Only then can we set

$$\alpha \approx \frac{i}{k R M_1} = \frac{i}{R(k U - \Omega)},$$

which enables us to pass to Formula (4.85).

The reason for the first requirement, namely that $M_1^2 \gg 1$, is obvious: the gas must move sufficiently fast relative to the waves propagated through the shell. The requirement that $M_1^2 \gg 1$ can be written in the form

$$M^2(1 - \bar{\omega})^2 \gg 1, \tag{4.87}$$

where

$$\bar{\omega} = \frac{V}{U} = \frac{\Omega}{k U}.$$

Since $1/k$ is a characteristic linear dimension of the disturbance, the parameter $\bar{\omega}$ can be treated as a reduced frequency. Condition (4.87) does not coincide with Lighthill's condition

$$M(\eta + \bar{\omega}) < 1,$$

but it is satisfied if, in addition to Lighthill's condition, the condition $M^2 \gg 1$ is satisfied.

The condition $k R M_1 \gg n$ is not included in any of the conditions listed above. It provides a relation between the variability characteristics of the disturbances in the direction of and at right angles to the flow. It must be assumed that analogous requirements are satisfied in other cases.

Comparison of the results given by simple formulae of the type (4.77) with the results of more exact aerodynamic theories[1] shows close agreement between the two starting with $M = 4$ and even with $M = 2$. With appropriate corrections depending on the value of $M^2 - 1$, the range of application of the formulae can be extended in the direction of smaller values of M (approximately as far as $M = \sqrt{2}$). The upper limit of applicability of the formulae has so far not been established. At very high supersonic velocities it is obviously essential to take into consideration the effects of viscosity, dissociation, ionization, phase changes on the boundary between the gas and the body, etc. However, locally the relation between the disturbances p and the inclination of the flow v must, of course, still hold. In particular, Formula (4.83) for small disturbances is still valid, although the coefficient in front of the brackets is a function of distance measured in the direction of the flow and depends on the behavior of the boundary layer of the undisturbed flow.

---

[1] See, for example, MORGAN, H. G., RUNYAN, H. L. and HUCKEL, V. Theoretical considerations of flutter at high Mach numbers. *J. Aeronaut. Sci.*, **25**, No. 6 (1958).

4.9. STABILITY OF ELASTIC PLATES AT HIGH SUPERSONIC VELOCITIES

Consider the small oscillations of a flat panel, rectangular in plan (Fig. 110), taking into account aerodynamic terms in accordance

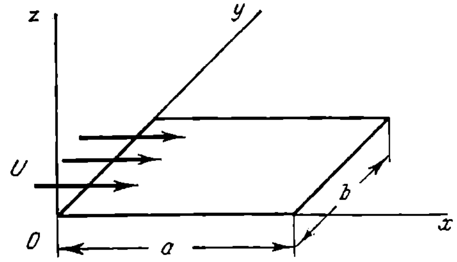


Fig. 110

with the linearized formula of the "law of plane sections" (4.83). In place of (4.67) we obtain a very much simplified equation

$$D \nabla^2 \nabla^2 w - \left(N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^2 w}{\partial y^2} \right) + \varrho_0 h \frac{\partial^2 w}{\partial t^2} + \\ + \varrho_0 h \varepsilon \frac{\partial w}{\partial t} + \frac{\varkappa p_\infty}{c_\infty} \left(\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} \right) = 0. \qquad (4.88)$$

Here all the notations of Section 4.7 still hold.

There are two very simple cases when the number of independent variables in Equation (4.88) can be reduced to two (namely x and t). These are the case of cylindrical bending and the case of a plate freely supported at the two sides parallel to the flow. In the latter case, by putting

$$w(x, y, t) = w_1(x, t) \sin \frac{n \pi y}{b} \qquad (n = 1, 2, \dots)$$

we can reduce Equation (4.88) to the form

$$D\left(\frac{\partial^4 w_1}{\partial x^4} - \frac{2n^2 \pi^2}{b^2} \frac{\partial^2 w_1}{\partial x^2} + \frac{n^4 \pi^4}{b^4} w_1\right) - \left(N_x \frac{\partial^2 w_1}{\partial x^2} - N_y \frac{n^2 \pi^2}{b^2} w_1\right) + + \varrho_0 h \frac{\partial^2 w_1}{\partial t^2} + \varrho_0 h \varepsilon \frac{\partial w_1}{\partial t} + \frac{\varkappa p_\infty}{c_\infty}\left(\frac{\partial w_1}{\partial t} + U \frac{\partial w_1}{\partial x}\right) = 0. \qquad (4.89)$$

The case of cylindrical bending can be derived from (4.89) by a limiting process $b \to \infty$.

We try to find a solution to Equation (4.89) in the form

$$w_1(x, t) = W(x)\, e^{st}, \qquad (4.90)$$

where $W(x)$ is some function which must satisfy the equation

$$\frac{d^4 W}{d\xi^4} - 2k\pi^2 \frac{d^2 W}{d\xi^2} + k^2 \pi^4 W + \beta \frac{dW}{d\xi} + \lambda W = 0. \qquad (4.91)$$

Here[1]

$$\xi = \frac{x}{a}, \qquad k = n^2 \varphi^2 + \frac{1}{2} \bar{N}_x, \qquad \varphi = \frac{a}{b},$$

$$\bar{N}_x = \frac{a^2 N_x}{\pi^2 D}, \qquad \bar{N}_y = \frac{a^2 N_y}{\pi^2 D}, \qquad \beta = \frac{\varkappa p_\infty U a^3}{D c_\infty},$$

$$\lambda = \omega^2 + \pi^4 d,$$

$$d = \frac{1}{4} \bar{N}_x^2 + n^2 \varphi^2 (\bar{N}_x - \bar{N}_y)$$

and also

$$\omega^2 = -(\sigma^2 + g\sigma), \qquad (4.92)$$

$$\sigma = \frac{s}{\Omega_0}, \qquad g = \frac{1}{\Omega_0}\left(\varepsilon + \frac{\varkappa p_\infty}{\varrho_0 h}\right), \qquad \Omega_0 = \frac{1}{a^2} \sqrt{\frac{D}{\varrho_0 h}}. \qquad (4.93)$$

The plane form of equilibrium of the plate is stable if all the characteristic exponents s lie in the left-hand half-plane of the complex variable. Comparing Equations (4.91) and (4.92) with Equations (2.77) and (2.79), we see that the present problem is very similar to the problem of the stability of the rectilinear form of a bar subjected to non-conservative torsion. The undisturbed form of the plate is stable provided that all the eigenvalues ω^2 of the boundary-value problem defined by Equation (4.91) and by the corresponding boundary conditions lie inside the "stability parabola", i.e. provided

---

[1] The reasons for splitting the term containing W into two will be seen from what follows.

$$\operatorname{Im}\omega^2 < g(\operatorname{Re}\omega^2)^{1/2}. \tag{4.94}$$

Thus the problem reduces to one of investigating the complex eigenvalues ω and applying Condition (4.94).

The laborious calculations can be reduced by replacing the stability condition (4.94) by the approximate condition (2.82) or by the sufficient condition that $\operatorname{Im}\omega^2 < 0$. The latter corresponds to solutions which do not take into account aerodynamic or structural damping, when the "stability parabola" degenerates into a narrow strip either side of the positive real semi-axis (Fig. 59). The question of the admissibility of ignoring damping in non-conservative problems of elastic stability has been discussed in Chapter 1; in Section (4.12) we shall return to this question and consider it from the point of view of panel flutter.

The boundary-value problem defined by Equation (4.91) and by the corresponding boundary conditions has been considered by a number of authors.[1] Setting

$$W(x) = e^{rx},$$

in Equation (4.91), we obtain the characteristic equation

$$(r^2 - k^2\pi^2)^2 + \beta r - \lambda = 0, \tag{4.95}$$

where $\lambda = \omega^2 + \pi^4 d$. Suppose that $r_1 = \mu + i\nu$ and $r_2 = \mu - i\nu$ are two complex conjugate roots of this equation. Then the other two roots, and also the parameters β and λ, can be expressed in terms of μ and ν as follows:

$$\left.\begin{aligned} r_{3,4} &= -\mu \pm \sqrt{(\nu^2 - 4\mu^2)}, \\ \beta &= -4\mu(\nu^2 - \mu^2 + k\pi^2), \\ \lambda &= k^2\pi^4 + (\mu^2 + \nu^2)(\nu^2 - 3\mu^2 + 2k\pi^2). \end{aligned}\right\} \tag{4.96}$$

---

[1] See reference 1 on p. 200. The remainder of this Section is based on the paper by Movchan.

Consider now the boundary-value problem corresponding to the conditions $W(0) = W''(0) = W(1) = W''(1) = 0$. Applying the boundary conditions to the solution, we obtain the equation

$$\Delta(k, \lambda, \beta) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ r_1^2 & r_2^2 & r_3^2 & r_4^2 \\ e^{-r_1} & e^{-r_2} & e^{-r_3} & e^{-r_4} \\ r_1^2 e^{-r_1} & r_2^2 e^{-r_2} & r_3^2 e^{-r_3} & r_4^2 e^{-r_4} \end{vmatrix} = 0,$$

which is extremely cumbersome to deal with. However, if we express the exponents r_k in terms of the variables μ and ν according to (4.96), we can reduce this equation to the simplified form

$$\Delta(k, \mu, \nu) = \mu^2[\cosh 2\mu - \cosh \sqrt{(\nu^2 - 2\mu^2 + 2k\pi^2)} \cos\nu] +$$
$$+ \frac{(\nu^2 - \mu^2 - k\pi^2)^2 + 2\mu^2(\mu^2 - k\pi^2)}{\sqrt{(\nu^2 - 2\mu^2 + 2k\pi^2)}} \frac{\sin\nu}{2\nu} \sinh \sqrt{(\nu^2 - 2\mu^2 + 2k\pi^2)} = 0.$$

Similarly, in the case of clamped leading and trailing edges, i.e. when $W(0) = W'(0) = W(1) = W'(1) = 0$[1], we obtain the equation

$$\Delta(k, \lambda, \beta) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ r_1 & r_2 & r_3 & r_4 \\ e^{-r_1} & e^{-r_2} & e^{-r_3} & e^{-r_4} \\ r_1 r^{-r_1} & r_2 e^{-r_2} & r_3 e^{-r_3} & r_4 e^{-r_4} \end{vmatrix} = 0,$$

which after rearrangement can be reduced to the form

$$\Delta(k, \mu, \nu) = \cosh 2\mu - \cosh \sqrt{(\nu^2 - 2\mu^2 + 2k\pi^2)} \cos\nu +$$
$$+ \frac{k\pi^2 - 3\mu^2}{\sqrt{(\nu^2 - 2\mu^2 + 2k\pi^2)}} \frac{\sin\nu}{\nu} \sinh \sqrt{(\nu^2 + 2\mu^2 + 2k\pi^2)} = 0.$$

The calculations are carried out as follows. By a method of successive approximations we find the roots μ and ν of the equation $\Delta(k, \mu, \nu) = 0$. Formulae (4.96) are then used to find the corresponding values of β and λ. If $\beta = 0$ the boundary-value problem is self-adjoint and all the eigenvalues λ are real. At some value of β one of the pairs of eigenvalues merges and then becomes complex adjoint (Fig. 111). Since $\lambda = \omega^2 + \pi^4 d$ and d is a real number, the value β_* corresponds to violation of the sufficient stability condition $\operatorname{Im}\omega^2 < 0$. The corresponding velocity can be provisionally called the critical flutter velocity, although, as can be seen from (4.92) for example, in order for flutter to occur the magnitude of $\operatorname{Im}\omega^2$ must exceed *in modulo* some quantity which depends on the degree of damping. Negative values of ω^2 correspond to divergent instability. The critical divergence velocity can therefore be found from the condition that $\lambda = \pi^4 d$. From the equation $\Delta(\lambda, \mu, \nu) = 0$ we can draw certain qualitative and quantitative conclusions concerning the influence of a number of factors on the stability of the plane form of the panel. In particular, it is found that there are points in the λ, β plane for which an exact solution can be found to Equation (4.91). These points correspond to a velocity parameter

---

[1] We recall that the edges parallel to the undisturbed flow are assumed to be simply supported.

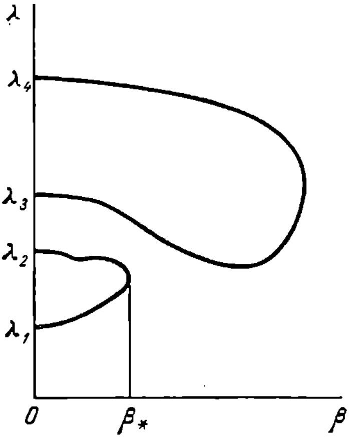


Fig. 111

$$\beta = \frac{8\pi^3 m_1}{3}(m_1^2 + m^2).$$

For a simply-supported plate the solution to Equation (4.91) is of the form

$$W(\xi) = \sin m\pi\xi \cdot \sin(m\pi\xi + \chi)\, e^{-m_1\pi\xi},$$

where m and n are positive integers, and the number m_1 and the phase χ are given by the formulae

$$m_1 = \frac{1}{\sqrt{3}}\left(2m^2 + n^2\varphi^2 + \frac{1}{2}\overline{N_x}\right)^{\frac{1}{2}} \qquad \chi = \arctan\frac{1}{m_1}.$$

Similarly, for a plate fully fixed at two edges we obtain

$$W(\xi) = \sin^2 m\pi\xi\, e^{-m_1\pi\xi}.$$

These exact results provide a useful basis of comparison for results obtained by approximate methods.

In contrast to plates, we arrive in the case of shells at differential equations of the eighth order with four boundary conditions to be satisfied on each side. A number of papers[1] dealing with panel flutter under conditions very similar to those outlined above have aimed at reducing the problem to a fourth order equation. However, we shall not study these papers here.

4.10. Application of Galerkin's Variational Method. Effect of Damping and of Forces in the Middle Surface

We have seen that by introducing aerodynamic terms on the basis of the "law of plane sections", the procedure is considerably simplified and in some cases leads to an exact solution of the problem. However, when the boundary conditions are complicated and account is taken of the true laws governing structural damping, especially in the case of non-linear problems, we find it necessary to resort to approximate methods of solution. The usual method is to express the required solution in an expansion in functions which satisfy the boundary conditions and then to apply Galerkin's variational method. In this way the problem is reduced to one of solving an equation obtained by equating to zero the corresponding system of homogeneous linear algebraic equations[2]. The present section will deal not so much with the calculations involved in the problem, as with a description of a number of interesting properties of flutter problems which can be readily understood if we pass to a finite system.

---

[1] Stepanov, R. D. On the flutter of cylindrical shells and panels moving in a flow of gas. *PMM*, **21**, No. 5 (1957); Movchan, A. A. On the effect of aerodynamic damping on supersonic skin flutter. *Izv. Akad. Nauk USSR, OTN, "Mekhanika i mashinostroenie"*, No. 1 (1960).

[2] Hedgepeth, J. Flutter of rectangular simply supported panels at high supersonic velocities. *J. Aeronaut. Sci.*, **24**, No. 8 (1957); Fung, Y. C. On the two-dimensional panel flutter. *J. Aeronaut. Sci.*, **25**, No. 3 (1958); Bolotin, V. V. On the question of the stability of a plate in a flow of compressible gas. *Sb. "Voprosy prochnosti materialov i konstruktsii"*. Izd-vo. Akad. Nauk USSR, 1959; Livanov, K. K. Stability of a clamped panel in a supersonic flow. *Inzh. sborn.*, **25** (1959), and others. Let us mention also the following publications: Makarov, B. P. On the stability of clamped plates in compressible gas flow, *Izv. Vyssh. uchebn. zaved. SSSR, Mashinostroenie*, No. 1 (1961); Rattayya, J. W. Flutter analysis of circular panels, *J. Aerospace Sci.*, **29**, No. 5 (1962) and others. The flutter of cylindrical shells of finite length was considered applying Galerkin's method in the following papers: Shveiko, Iu. Iu. Stability of a circular cylindrical shell in a gas flow, *Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie*, No. 6 (1960); Holt, M., Strack, S. L. Supersonic panel flutter of a cylindrical shell of finite length, *J. Aerospace Sci.*, **28**, No. 3 (1961); Voss, H. M. The effect of an external supersonic flow on the vibration characteristics of thin cylindrical shells, *J. Aerospace Sci.*, **28**, No. 12 (1961).

We try to find a solution to Equation (4.88) in the form of a series

$$w(x, y, t) = \sum_{k=j}^{\infty} f_k(t)\, w_k(x, y)$$

in functions $w_k(x, y)$ which satisfy the boundary conditions for $w(x, y, t)$. Applying Galerkin's method, we obtain the system of equations

$$\frac{d^2 f_j}{dt^2} + (\varepsilon + \gamma)\frac{df_j}{dt} + \sum_{k=j}^{\infty} a_{jk} f_k + \frac{\varkappa\, p_\infty\, U}{c_\infty} \sum_{k=j}^{\infty} b_{jk} f_k = 0$$

$$(j = 1, 2, \ldots),$$

where

$$a_{jk} = \frac{\int\limits_0^a\int\limits_0^b \left[D\, \nabla^2\, \nabla^2\, w_k - \left(N_x \frac{\partial^2 w_k}{\partial x^2} + N_y \frac{\partial^2 w_k}{\partial y^2}\right)\right] w_j\, dx\, dy}{\int\limits_0^a\int\limits_0^b \varrho_0\, h\, w_j\, w_k\, dx\, dy}$$

$$b_{jk} = \frac{\int\limits_0^a\int\limits_0^b \frac{\partial w_k}{\partial x} w_j\, dx\, dy}{\int\limits_0^a\int\limits_0^b \varrho_0\, h\, w_j\, w_k\, dx\, dy}, \qquad \gamma = \frac{\varkappa\, p_\infty}{\varrho_0\, h\, c_\infty} = \frac{\varrho_\infty\, c_\infty}{\varrho_0\, h}.$$

If for the function $w_k(x, y)$ we select the natural modes of the plate *in vacuo*, we have

$$a_{jk} = \Omega_j^2\, \delta_{jk},$$

where Ω_j are the corresponding natural frequencies. After introducing the non-dimensional parameters

$$\Omega_0\, t = \tau, \qquad \omega_j = \frac{\Omega_i}{\Omega_0}, \qquad g = \frac{\varepsilon + \gamma}{\Omega_0},$$

$$\beta = \frac{4\varkappa\, p_\infty\, M}{\varrho_0\, h\, a\, \Omega_0^2}, \qquad \bar{b}_{jk} = \frac{\varrho_0\, h\, a}{4} b_{jk}$$

(Ω_0 is a characteristic frequency, for example $\Omega_0 = \Omega_1$) we can write the system of equations in the form

$$\frac{d^2 f_j}{d\tau^2} + g\frac{df_j}{d\tau} + \omega_j^2 f_j + \beta \sum_{k=1}^{\infty} \bar{b}_{jk} f_k = 0 \qquad (4.97)$$

$$(j = 1, 2, \ldots).$$

The plane form of the bar is stable if all the roots of the equation

$$|(\omega_j^2 + \sigma^2 + g\,\sigma)\,\delta_{jk} + \beta\,\bar{b}_{jk}| = 0 \qquad (4.98)$$

lie in the left-hand half-plane. If the effect of damping is not taken into account, the stability condition reduces to the requirement that all the frequencies $w = i\,\sigma$ given by the equation

$$|(\omega_j^2 - \omega^2)\,\delta_{jk} + \beta\,\bar{b}_{jk}| = 0, \qquad (4.99)$$

remain on the real axis. The parameter β corresponding to divergence conditions can be found from the equation

$$|\omega_j^2\,\delta_{jk} + \beta\,\bar{b}_{jk}| = 0. \qquad (4.100)$$

Consider the case of a simply supported panel.[1] In this case, for a fixed number of half-waves in the lateral direction, we have

$$w_j = \sin\frac{j\pi x}{a}\sin\frac{n\pi y}{b},$$

$$\omega_j^2 = (j^2 + n^2\varphi^2)^2 + j^2\bar{N}_x + n^2\varphi^2\bar{N}_y, \qquad (4.101)$$

where the notations of Section (4.9) still hold. Carrying out the necessary calculations, we find that

$$\bar{b}_{jk} = \left\{\begin{array}{lll} \dfrac{jk}{j^2 - k^2}, & \text{if } j \pm k & \text{is an even number} \\ 0, & \text{if } j \pm k & \text{is an odd number} \end{array}\right\} \qquad (4.102)$$

Note that the reciprocity condition (4.75), which indicates the non-conservative nature of the problem, is also fulfilled. In order to determine the critical parameter β_* we use the results already obtained in Sections 1.14–1.16. Taking as a first approximation a second-order determinant, we obtain the formula

$$\beta_{**} = \frac{3}{4}\sqrt{[(\omega_2^2 - \omega_1^2)^2 + 2g^2(\omega_1^2 + \omega_2^2)]}, \qquad (4.103)$$

---

[1] This problem without the effect of damping has been investigated in a number of papers. See, for example, Johns, B. J. Some flutter studies using piston theory. *J. Aeronaut. Sci.*, 25, No. 11 (1958).

which is completely analogous to Formula (1.96). If the partial natural frequencies Ω_1 and Ω_2 are sufficiently far apart, the correction to the damping introduced by the second term under the root sign is negligibly small. In this case we have the approximate relation

$$\beta_* = \frac{3}{4}(\omega_2^2 - \omega_1^2). \tag{4.104}$$

Formulae (4.103) and (4.104) give an accurate estimate of the order of the critical parameter. Thus, for a plate of infinite span with no applied axial forces ($N_x = 0, \omega_j = j^2$) Formula (4.104) gives $\beta_* = 11.25$, whereas an exact solution gives $\beta_* = 13.9$ (if four terms of the series are taken the formula gives $\beta_* = 13.6$). In spite of their simplicity Formulae (4.103) and (4.104) take into account all the main characteristics of the problem of panel flutter.

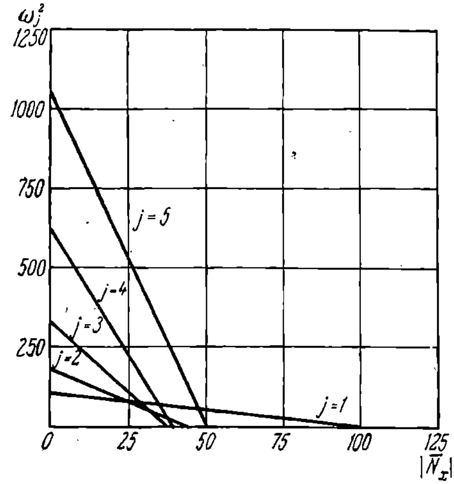


Fig. 112

It has been pointed out already that damping only becomes significant when the partial frequencies are close together. For heated panels elongated in the direction of the flow multiple frequencies are quite likely. Figure 112 shows the relation between the square of the non-dimensional frequency and the non-dimensional compressive force for a panel with the ratio of its sides $\varphi = a/b = 3.0$, $N_y = 0$. A typical relation between the critical parameter β_* and $|\bar{N}_x|$ is shown in Fig. 113. It will be seen that β_* has minima which are close to values of $|\bar{N}_x|$ corresponding to the multiple frequencies and which are of the order of g. It will be seen also that the critical value of the parameter does not necessarily correspond to the pair of smallest frequencies: the important thing here is not so much the magnitude of the frequencies, but the interval between adjacent frequencies (Fig. 114).

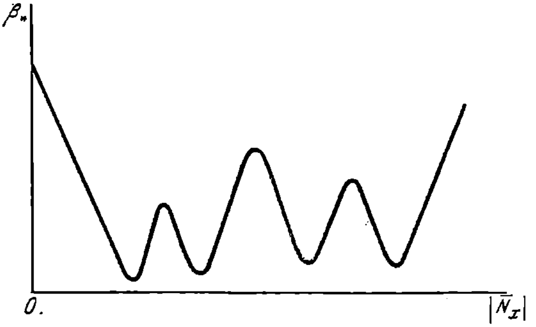


Fig. 113

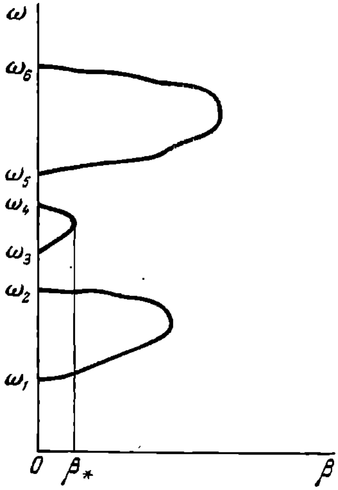


Fig. 114

So far we have assumed that the partial damping coefficients are the same. However, it has already been pointed out (Section 2.5) that there is no reason for assuming that the coefficients of structural damping are equal (aerodynamic damping is the same for all degrees of freedom). Instead of (4.98) consider the equation

$$|(\omega_j^2 + \sigma^2 + g_j\,\sigma)\,\delta_{jk} + \beta\,\bar{b}_{jk}| = 0,$$

in which

$$g_j = \frac{\varepsilon_j + \gamma}{\Omega_0},$$

or alternatively

$$g_j = \frac{\gamma}{\Omega_0}(1 + G_j).$$

Here, in the same way as (4.66),

$$G_j = \frac{\psi_j\,\omega_j^2}{2\pi\,\omega}\,\frac{\varrho_0}{\varrho_\infty}\,\frac{\Omega_0\,h}{c_\infty},$$

ψ_j is the relative dissipation of energy due to structural damping corresponding to the j-th mode of oscillation; the formula is valid for periodic processes with a non-dimensional frequency ω. Putting

$$\Omega_0 = \frac{\pi^2}{a^2}\sqrt{\frac{D}{\varrho_0\,h}}, \qquad D = \frac{E\,h^3}{12(1-\mu^2)}, \qquad c_\infty = \sqrt{\frac{\varkappa\,p_\infty}{\varrho_\infty}},$$

we obtain the following formula for G_j:

$$G_j = G_0\frac{\psi_j\,\omega_j^2}{\omega}, \qquad G_0 = \frac{\pi}{\sqrt{[48(1-\mu^2)]}}\left(\frac{h}{a}\right)^2\left(\frac{E}{\varkappa\,p_\infty}\,\frac{\varrho_0}{\varrho_\infty}\right)^{1/2}.$$

For a steel plate in air under normal conditions

$$G_0 \approx 12.6 \times 10^4\left(\frac{h}{a}\right)^2.$$

For a square plate without loading $\omega_1 = 2$, $\omega_2 = 5$ and the non-dimensional frequency on the flutter boundary is

$$\omega_* = \sqrt{\frac{\omega_1^2 + \omega_2^2}{2}} \approx 3.81.$$

If $h/a = 0.01$, then even if $\psi = 0.10$ (a comparatively small structural damping) we find that $G_1 = 1.32$ and $G_2 = 8.25$. Thus the effect of structural damping is no less significant than that of aerodynamic damping.

It was shown in Section (1.16) that if the partial coefficients of damping ε_j (or g_j) are different, the critical value of the non-conservative force parameter β_{**}, when damping is taken into account, can be less than the value of β_* for a system without damping. This still holds for flutter problems. It will be seen that Condition (4.94) is valid only for the case when the damping coefficients coincide; therefore the sufficient condition of stability $\operatorname{Im}\omega^2 < 0$ and the resulting condition $\beta_{**} \geqq \beta_*$ are valid only for this case.

The coefficients of structural damping are, in general, different for different modes of oscillation. This might account for the decrease in the parameter β_{**} compared with β_*. This effect is somewhat reduced by the fact that the coefficients of aerodynamic damping are equal.

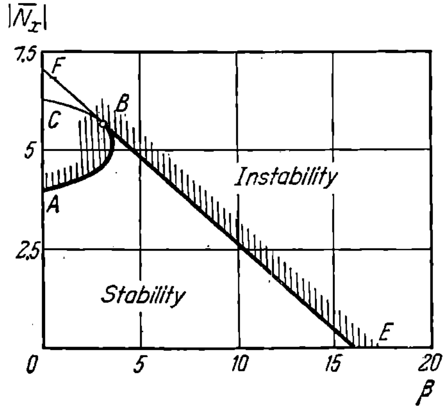


Fig. 115

Making the quite reasonable assumption that the relative dissipation of energy due to structural damping is the same for all modes, we find that

$$\eta = \frac{1 + G_0 \dfrac{\psi\,\omega_2^2}{\omega_*}}{1 + G_0 \dfrac{\psi\,\omega_1^2}{\omega_*}}.$$

Here $\eta = g_2/g_1$ is the parameter appearing in the approximate formula (1.98):

$$\beta_{**} \approx \beta_* \frac{2\sqrt{\eta}}{1 + \eta}.$$

As an example consider a square plate without loading ($\omega_1 = 2.00$, $\omega_2 = 5.00$). Letting the parameter $G_0\,\psi$ vary from zero to 5.00, we find that β_{**}/β_* varies from 1.00 to 0.73. For $h/a = 0.01$, this corresponds to a damping decrement of approximately 0.2.

On the basis of the general theory (Section 1.15), we can expect the addition of aerodynamic forces under certain conditions to contribute to the stability of a plate compressed by axial forces. Figure 115 shows the region of stability in the plane of β, $|\bar{N}_x|$ with $\varphi = 1$ and $\bar{N}_y = 0$. If in Equation (4.100) we retain the second-order determinant, we obtain the following formula for the divergence parameter:

$$\beta_* = \frac{3}{2}\sqrt{(\omega_1 \omega_2)}.$$

At the same time the flutter parameter is given by Formula (4.104). The lower branch of the curve ABC corresponds to the buckling boundary, and the straight line BE corresponds to the flutter boundary. With increase in the velocity of flow parameter from zero the aerodynamic forces slightly increase the stability of the plate against buckling over a certain interval.

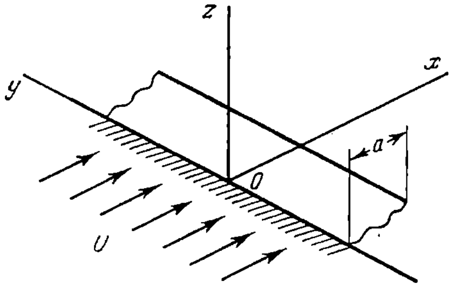


Fig. 116

In conclusion, we shall consider briefly the problem of the stability of a plate clamped at one edge and free at the opposite edge, and extending to infinity in the direction perpendicular to the flow (Fig. 116). This problem is rather interesting in that when the flow is in a direction from the clamped edge to the simply supported edge, flutter is possible, and when the flow is in the opposite direction, divergence is possible.

We shall consider this problem on the assumption of cylindrical bending without taking into account axial forces and damping. Taking as an approximation the beam functions

$$w_j(\xi) = \cosh\lambda_j\xi - \cos\lambda_j\xi - \frac{\cosh\lambda_j + \cos\lambda_j}{\sinh\lambda_j + \sin\lambda_j}(\sinh\lambda_j\xi - \sin\lambda_j\xi),$$

$$\omega_j = \lambda_j^2, \qquad \lambda_1 = 1.875, \qquad \lambda_2 = 4.694, \ \ldots$$

and noting that

$$\int_0^1 w_j w_k' \, d\xi = \begin{cases} \dfrac{[w_j''(0)]^2}{2\lambda_j^4} & \text{if } j = k, \\[2mm] \dfrac{1}{\lambda_j^4 - \lambda_k^4}[-w_j(1)\, w_k(1)\, \lambda_j^4 + w_j''(0)\, w_k''(0)] & \\ & \text{if } j \neq k, \end{cases}$$

$$\int_0^1 w_j w_k \, d\xi = \frac{1}{4} w_j^2(0)\, \delta_{jk},$$

we find $\bar{b}_{11} = 2.000$, $\bar{b}_{22} = 2.000$, $\bar{b}_{12} = -3.450$, $\bar{b}_{21} = 1.055$. Thus $\bar{b}_{12}\bar{b}_{21} < 0$, although $\bar{b}_{12} \neq -\bar{b}_{21}$. If we limit ourselves to the second order determinant, we obtain a simple equation for the non-dimensional frequency ω as a function of the velocity parameter[1] β'. The results of the calculations are plotted in Fig. 117. If $\beta < 0$, i.e. if the flow is

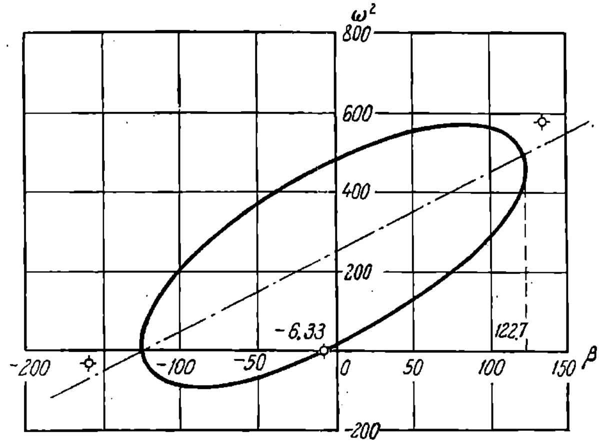


Fig. 117

---

[1] Here we have taken.

$$\Omega_0^2 = \Omega_1^2 = \frac{\lambda_1^4}{a^2}\sqrt{\frac{D}{\varrho_0 h}}.$$

in a direction from the free edge to the clamped edge, then as the velocity increases, divergence occurs first (when $\beta_* = 6.33$). If $\beta > 0$, divergence is not possible and flutter occurs at $\beta_* = 122.7$. It should be pointed out that an exact solution of this problem[1] gives a value for the divergence parameter which is practically identical with that quoted above. In Fig. 117 two further points are given; these refer to the exact solution.

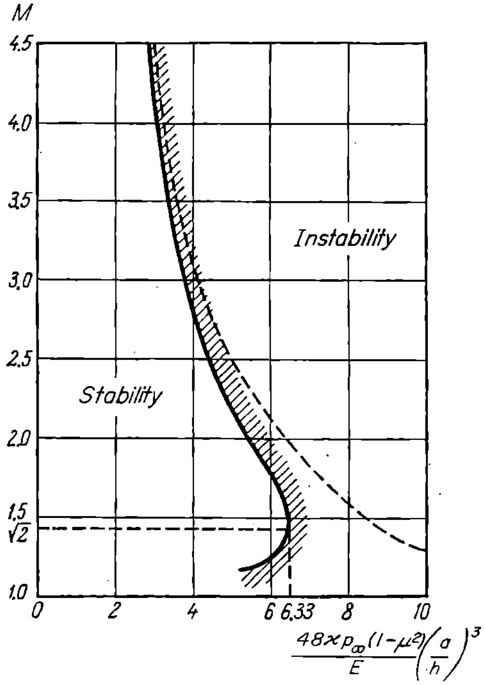


Fig. 118

If instead of Formula (4.83) we use the formula

$$\bar{p} = \frac{\varrho_\infty U}{\sqrt{(M^2 - 1)}}\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right),$$

we can somewhat extend the field of application of the results. Figure 118 shows the relation between the critical Mach number for divergence and the parameter

---

[1] Movchan, A. A. On the oscillations of plates moving in a gas. *PMM*, 20, No. 2 (1956).

$$\beta_0 = \frac{48\varkappa p_\infty(1 - \mu^2)}{E}\left(\frac{a}{h}\right)^3.$$

The broken line gives the results obtained from the usual formula for the pressure (4.83). The smallest relative thickness necessary to ensure stability occurs approximately at $M = \sqrt{2}$, which agrees qualitatively with the result obtained for a plate exposed to a potential flow under different support conditions (Fig. 106). The solution is invalid at transonic velocities.

4.11. LIMITS OF APPLICATION OF GALERKIN'S METHOD. EXPLANATION OF A PARADOX IN THE PROBLEM OF MEMBRANE FLUTTER

Problems of panel flutter belong to a class of non-self-adjoint boundary-value problems, the general properties of which are as yet not fully understood. In this connection doubts have often been expressed as to the admissibility of Galerkin's method as applied to problems of this sort. The basis of these doubts may be explained as follows.

The linear problem of the stability of a rectangular membrane in a flow of gas can be solved without difficulty, and its solution shows that for a membrane fixed along its contour no critical flutter velocity exists, i.e. the plane form of a stretched membrane is stable at all sufficiently high supersonic velocities. It would appear, however, that the application of Galerkin's method to this problem gives a critical flutter velocity which tends to some limiting value as the number of terms that are taken increases. On this basis the assumption has been made[1] that Galerkin's method gives an incorrect answer to the question of critical flutter velocity, not only for a membrane, but also for a plate, and that cases of plate failure observed must be explained by other causes.

On the other hand, in the case of plates there is close agreement between the results of exact analyses and those based on Galerkin's method, provided that in the latter method four terms of the series are retained. In this connection the question arises as to the reason for the significant difference in the results given by Galerkin's method when applied to plates and when applied to membranes (i.e. to plates the cylindrical stiffness of which tends to zero). This question has not only a theoretical, but also a practical significance. The majority of problems of panel flutter, and particularly non-linear problems, have to be solved by resorting to Galerkin's variational method. It is therefore extremely important to know the limits of application of the method to problems of this type. The paradoxical divergence in the results for a membrane has until now remained unexplained, and so the major part of this section will be devoted to an explanation of this divergence.[1]

---

[1] ASHLEY, H. and ZARTARIAN, C. Piston theory—a new aerodynamic tool for the aeroelastician. *J. Aeronaut. Sci.*, 23, No. 6 (1956); BISPLINGHOFF, R. L. Some structural and aeroelastic considerations of high-speed flight. *J. Aeronaut. Sci.*, 23, No. 4 (1956); see also the book: BISPLINGHOFF, R. L., ASHLEY, H. and HALFMAN, R. L. *Aeroelasticity* Addison-Wesley 1955.

Consider once more a flat panel, rectangular in plan with sides a and b, one side of which is exposed to a supersonic flow with an undisturbed velocity U (see Fig. 110). We shall ignore damping and confine our attention to the plane problem, for which Equation (4.88), after the substitution

$$w(x, t) = W(x)\, e^{i\Omega t}$$

becomes

$$D\frac{d^4 W}{dx^4} - N_x\frac{d^2 W}{dx^2} - \varrho_0 h\,\Omega^2 W + \varkappa\,\varrho_\infty M\frac{\partial W}{\partial x} = 0. \qquad (4.105)$$

It can easily be shown that for a membrane ($D = 0$) all the frequencies are real irrespective of the Mach number M, i.e. the plane form of the membrane remains stable at any sufficiently high supersonic velocities. In fact, a calculation leads to the formula

$$\Omega = \frac{\pi}{a}\sqrt{\frac{N_x}{\varrho_0 h}}\sqrt{\left[m^2 + \left(\frac{\varkappa\, p_\infty M a}{2\pi N_x}\right)^2\right]}, \qquad (4.106)$$

where m is any positive integer.

If we now apply Galerkin's method to this problem, expressing the solution in the form of a series

$$W(x) = \sum_{k=1}^{\infty} f_k \sin\frac{k\pi x}{a}, \qquad (4.107)$$

---

[1] Bolotin, V. V. On the application of Galerkin's variational method to problems of flutter of elastic panels. *Izv. vyssh. shkoly., ser. "Mashinostroenie"*, No. 11 (1959).

then for a plate, even to a first approximation (when two terms of the series are retained), we obtain results which are fairly close to the exact solution. In the case of a membrane ($D = 0$), again to a first approximation, we obtain $\beta_* = 2.25$. At the same time, an exact solution shows that membrane flutter in general does not occur.

In order to clarify the way in which β_* behaves with increase in the number of terms in the series (4.107) and to establish the reason for the qualitative difference between exact and approximate solutions, calculations of the frequencies ω have been carried out for various numbers of terms. The resulting curves for a plate with $N_x = 0$ are shown in Figs. 119–122. Closure of a "loop" indicates two complex conjugate frequencies, and the minimum value of the parameter β_* at which closure occurs is the critical value. It will be seen from the graphs that the critical value of β_* varies only slightly with increase in the number of terms in the series (for two, three, four and five terms we obtain $\beta_* = 11.2$, 14.3, 13.6 and 13.8, respectively). It is typical in the case of an odd number of terms for one of the loops to open up. This is due to the absence of a "coupling" coordinate for one of the generalized coordinates. In general, an approximation with an even number of terms of Series (4.107) is more suitable for a problem in which instability is associated with the "coupled" interaction of generalized coordinates.

The curves for a membrane are shown in Figs. 123–126. It should be noted that the basic loop corresponding to the first two terms of Series (4. 107) changes in form with increase in the number of terms, and that for an odd number it opens up. At the same time the "loop" corresponding to the last two terms is practically unaltered and closes at approximately the same value, $\beta_* = 2.3$. It appears that so far everybody who has applied Galerkin's method to the problem of a membrane has calculated just this minimum value and taken it to be the critical value, whereas the last terms taken should have been looked upon simply as a correction to the basic solution, without taking into account results which are obtained principally from these terms. If in fact the behavior of the first "loop" is considered, it will be found that there is no real contradiction between the exact solution and a solution by Galerkin's method. Two terms of the series give $\beta_* = 2.25$, three terms give $\beta \to \infty$ opening up of the first "loop", four terms give $\beta_* = 3.22$ and for five terms we once more obtain $\beta_* \to \infty$. We have here a typical divergent process for finding β_*. It should be noted that for the frequencies ω, at any rate, for sufficiently small

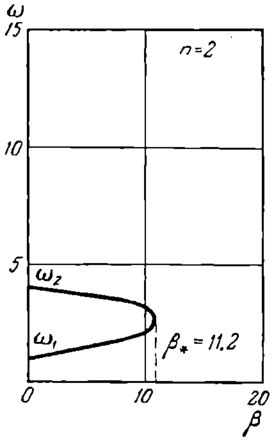


Fig. 119

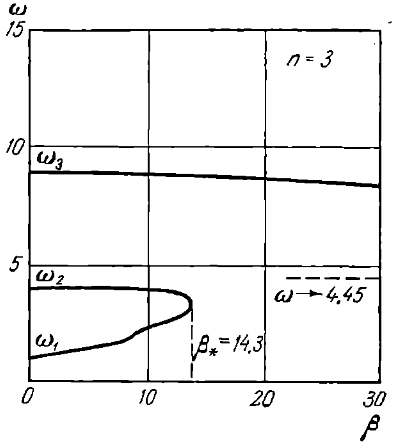


Fig. 120

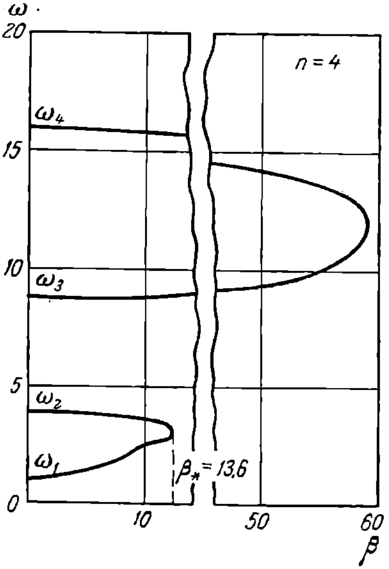


Fig. 121

values of β, the process of successive approximations does converge. This can be seen, for example, from Fig. 126, where the dotted lines represent the relations

$$\omega = \sqrt{\left(m^2 + \frac{\pi^2}{64}\beta^2\right)} \qquad (m = 1, 2, \ldots),$$

which are derived from the exact Formula (4.106)

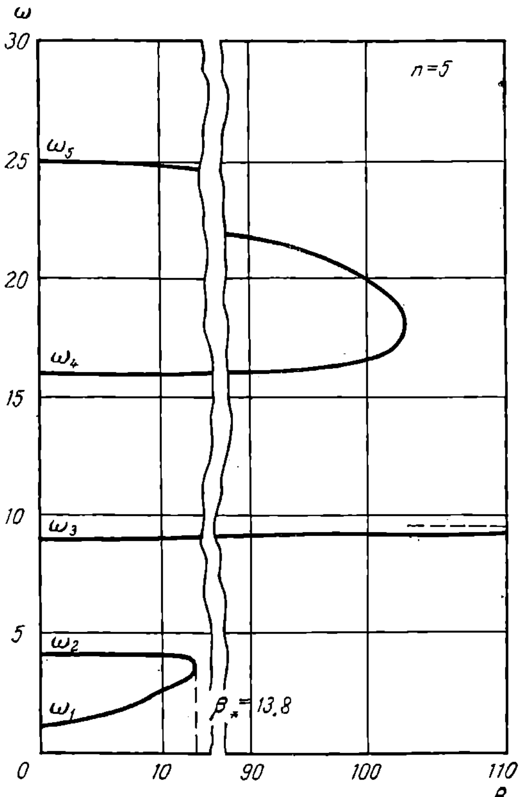


Fig. 122

Consider now the convergence of the infinite determinant (4.99). If we divide the jth row by ω_j and the kth column by ω_k we can express this determinant in the form

$$\Delta = |\delta_{jk} + c_{jk}|, \tag{4.108}$$

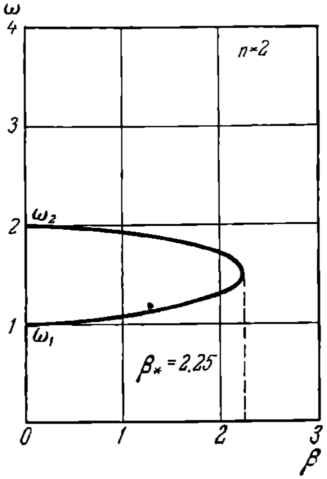


Fig. 123

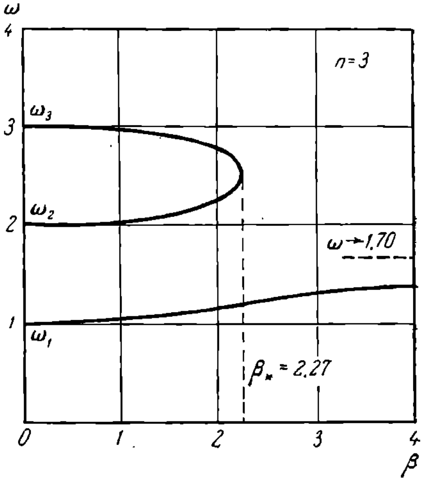


Fig. 124

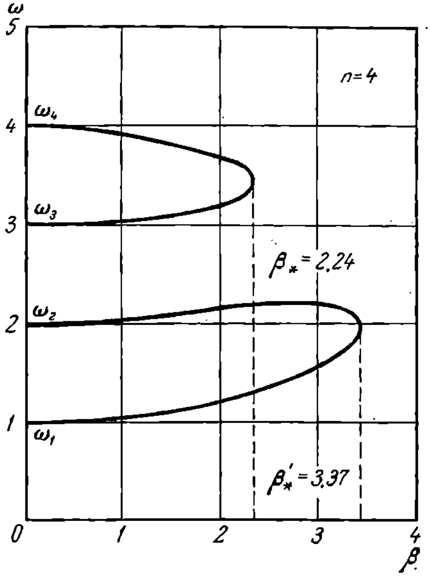


Fig. 125

where δ_{jk} is the Kronecker delta,

$$\left.\begin{aligned} c_{jk} &= \frac{\omega^2 \delta_{jk}}{\omega_j \omega_k} + \frac{jk}{\omega_j \omega_k} \frac{\beta}{j^2 - k^2} \quad (j \pm k \text{ is an odd number}), \\ c_{jk} &= \frac{\omega^2 \delta_{jk}}{\omega_j \omega_k} \quad (j \pm k \text{ is an even number}). \end{aligned}\right\} \tag{4.109}$$

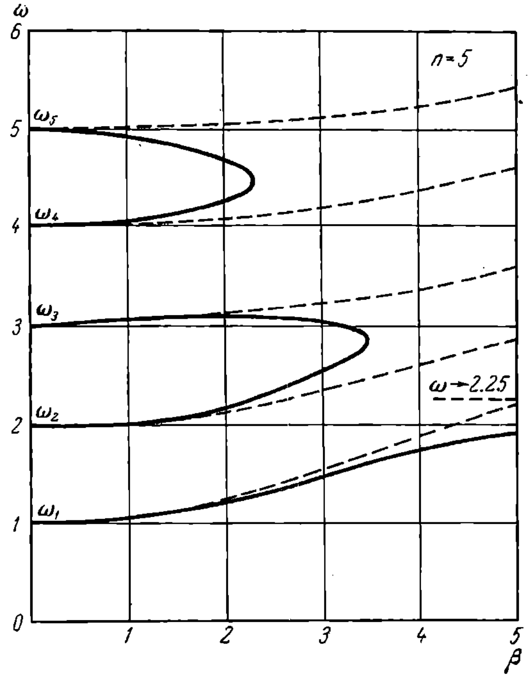


Fig. 126

It can be shown[1] that the infinite determinant (4.108) converges if the double series

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |c_{jk}| \tag{4.110}$$

converges. A determinant is described as normal when it satisfies this condition.

---

[1] Kantorovich, L. V. and Krylov, V. I. Approximate Methods of Higher Analysis (*Priblizhennye metody vysshego analiza*). Gostekhizdat, 1952.

From the corresponding formula of (4.101) with $n = 0$, $N_x = N_y = 0$, we have $\omega_k = k^2$. Since $|j^2 - k^2| \geqq j + k > 2\sqrt{(j\,k)}$ it follows that

$$|c_{jk}| < \frac{\omega^2\,\delta_{jk}}{j^4} + \frac{\beta}{2j^{3/2}\,k^{3/2}}.$$

The double series composed of the elements an the right-hand side of this expression converge, which proves the convergence of the series (4.110). Consequently, the determinant (4.108) for a plate is a convergent (normal) determinant. A similar result can be proved for a plate with other types of edge support.

Consider now the case of a membrane ($\omega_k = k$). Here the convergence of the series (4.110) depends on that of the series

$$\sum_j^\infty \sum_k^\infty \frac{1}{|j^2 - k^2|} = \sum_{j=1}^\infty \sum_{k<j}^\infty \frac{2}{j^2 - k^2}$$

(j and k are numbers of opposite evenness). Noting that if $k < j$

$$\frac{2}{j^2 - k^2} > \frac{2}{j^2}$$

and that the number of non-zero terms in the jth row is equal either to $j/2$ or to $(j-1)/2$, but in any case is not less than $j/3$, we obtain

$$\sum_j^\infty \sum_k^\infty \frac{1}{|j^2 - k^2|} > \sum_{j=1}^\infty \frac{2}{3j}.$$

The right-hand side is an obviously divergent series, which shows that the series (4.110) for a membrane diverges. Consequently the determinant (4.108) in the case of a membrane is not a normal determinant.

In the general case of a rectangular plate,

$$\omega_k^2 = (k^2 + n^2\,\varphi^2)^2 + \frac{a^2}{\pi^2 D}(k^2\,N_x + n^2\,\varphi^2\,N_y),$$

where $\varphi = a/b$. It is clear that in the case of a non-zero cylindrical stiffness D and with n fixed, we can always choose a finite number k so that the inequality $\omega_k^2 > C\,k^{7/2}$ is satisfied, where C is some constant. Then for sufficiently large j and k we have the inequality

$$|c_{jk}| < \frac{\omega^2\,\delta_{jk}}{C\,j^{7/2}} + \frac{\beta}{C\,j^{3/4}\,k^{3/4}\,|j^2 - k^2|} < \frac{\omega^2\,\delta_{jk}}{C\,j^{7/2}} + \frac{\beta}{2C\,j^{5/4}\,k^{5/4}}$$

and consequently the series (4.110) converges, provided $D \neq 0$. The convergence is slower the smaller D is, i.e. the more closely the plate approximates to a membrane.

From the foregoing analysis we see that the application of Galerkin's method to the problem of plate flutter leads to infinite determinants which belong to the class of normal (convergent) determinants. In the case of a membrane (a plate with a zero bending stiffness) the determinant is not normal and its use leads to an apparent inconsistency between the exact solution and one based on Galerkin's method. This inconsistency can be obviated by observing the behavior of the solution with gradual increase in the number of terms in the series. The ineffectiveness of Galerkin's method in the case of a membrane is not associated directly with the fact that the problem of panel flutter is non-self-adjoint; the same thing occurs in certain self-adjoint boundary-value problems presented in the appropriate manner. Although for the given class of problems a strict proof of the convergence of Galerkin's method does not follow from proof of the convergence of the determinants, the foregoing discussion has shown that in its application to plates with a non-zero bending stiffness this method gives extremely reliable results.

4.12. NON-LINEAR PROBLEMS IN THE THEORY OF AEROELASTICITY. EFFECT OF GEOMETRIC AND AERODYNAMIC NON-LINEARITIES

So far we have investigated the stability of plates and shells in a flow of gas according to the linear theory. By using the linear theory we can find the smallest value of the velocity (the critical flutter or divergence velocity) at which the undisturbed form of a flat or curved panel ceases to be stable. The analysis of non-linear problems is of interest in two respects. In the first place, in a number of cases when the critical flutter velocity is exceeded, this does not mean an immediate failure of the panel; this occurs only after a definite period of time and is a fatigue type of failure. In order to determine the expected useful life of a panel it is essential to determine its amplitude of oscillations in the flutter region, and this can be done only on the basis of the non-linear theory. Secondly, as the Mach number increases the effect of aerodynamic non-linearities becomes more prononced. In some problems the effect of these non-linearities is such that by taking them into account we are able to reveal the likelihood of undamped oscillations or divergent modes of equilibrium at velocities less than the critical flutter (or divergence) velocity, if a large enough disturbance is imparted to the system. In such cases the critical velocity determined by the methods of the linear theory of aeroelasticity is only the upper limit of the critical velocities for actual structures.

A most important non-linear factor which limits the amplitudes in the case of flutter of plates and shells, and their deflections in the case of static instability, is non-linearity of a geometric origin associated mainly with the occurrence of tensile stresses in the middle surface. These stresses depend very much on the boundary conditions, i.e. on the joint response of the plate or shell and the supporting frame or the structure as a whole. In addition to these factors, in a number of problems, it is necessary to take into account non-linear non-elastic effects and the effect of structural non-linearities (cutouts, arresting devices, etc.).

Let us consider in more detail the effect of aerodynamic non-linearity.[1] We can write Formula (4.77) in the form

$$p = p_\infty \left(1 + \frac{\varkappa - 1}{2} M \eta\right)^{\frac{2\varkappa}{\varkappa - 1}},$$

where η is a non-dimensional disturbance parameter (the angle of attack, the non-dimensional amplitude, etc.). One of the conditions for the applicability of the formula requires that $M\eta < 1$. However, in order to be able to make various approximations in the formula

$$p = p_\infty \left[1 + \varkappa M \eta + \frac{\varkappa(\varkappa + 1)}{4} (M\eta)^2 + \frac{\varkappa(\varkappa + 1)^2}{12} (M\eta)^3 + \cdots\right] \tag{4.111}$$

we need to impose even more stringent requirements. Thus the linear approximation gives satisfactory results provided $(M\eta)^2 \ll 1$. The greater the Mach number the narrower is the range of values $-\eta_1 < \eta < \eta_1$ for which reduction to the linear approximation is admissible.

Since we are concerned with stability "in the small", the equations can be linearized with respect to the elastic displacements of the body. However, only when the disturbances introduced by the rigid body satisfy the condition that $-\eta_1 < \eta < \eta_1$ (for example, in the case of motion of a body of zero thickness and zero angle of attack), does the result of linearizing the equations coincide with Formula (4.83)

$$\bar{p} = p_\infty \varkappa M \eta.$$

---

[1] Bolotin, V. V. On critical velocities in non-linear problems of the theory of aeroelasticity. *Nauchn. dokl. vyssh. shkoly*, "*Mashinostroenie i priborostroenie*", No. 3 (1958).

Thus, at high supersonic velocities the aerodynamic forces are for the most part non-linear. We see from Formula (4.111) that in many cases when $M \gg 1$ the aerodynamic forces have hard non-linearity in the sense that the sign of the non-linear contribution coincides with the sign of η. For instance, if a plane profile is exposed to a steady flow (Fig. 127), then in a calculation of the lift and aerodynamic moment the second order terms in Formula (4.111) disappear, and the non-linear contribution, therefore, starts with terms of the order of η^3 (Fig. 128). If one side only of the panel is exposed to the flow, then the quadratic terms, of course, are retained. However, in the case of motion of the panel close to harmonic oscillation at a frequency $\Omega (\eta \sim \eta_0 \sin \Omega t)$, the quadratic terms contain no components with this frequency, whereas the cubic terms do contain such components $(\eta^3 \sim 3/4 \eta_0^3 \sin \Omega t + \cdots)$. Therefore, here again the important non-linear contribution starts with terms of the order of η^3.

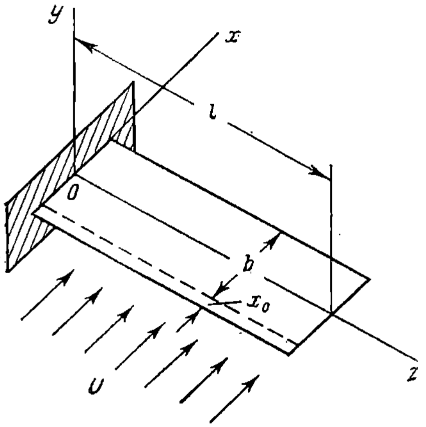


Fig. 127

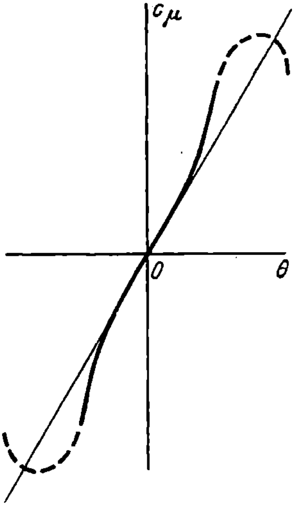


Fig. 128

Whenever the aerodynamic forces have hard non-linearity, their effect on the behavior of an elastic system will be essentially opposite to the effect of geometrical non-linearity. An increase in geometrical non-linearity (for example, an increase in the non-linear components of the stresses in the middle surface caused by the displacements being restrained at the edges of a plate or shell) causes a reduction in amplitude in the case of steady flutter. Conversely, an increase in aerodynamic non-linearity results in an increase in these amplitudes. Whenever there is considerable geometric non-linearity, the excitation of flutter or divergence buckling on the boundary of the region of instability will be "soft", in the sense that if the critical velocity is slightly exceeded, the amplitudes of steady flutter or the divergence deformations will be only small. In other words the stability boundary will be "safe" in the sense of Bautin[1]. Conversely, when aerodynamic non-linearity predominates, excitation of oscillations on the boundary of the instability region can be "hard", and the boundary of the stability region "dangerous". In this case the aerodynamic forces can maintain states of motion other than the undisturbed state, even when the undisturbed motion is stable "in the small". With decrease in the velocity U, emergence from the region of instability is accompanied by "entrainment". If this does in fact occur, the critical velocities determined from the linearized equations act as upper critical velocities. In addition to these velocities, the lower critical velocities assume importance, i.e. the lowest velocities at which excitation of steady flutter motion or divergent forms of equilibrium first become possible. The true critical velocity for a real structure in these cases depends very much on the nature of the disturbances (initial inaccuracies, initial angle of attack, atmospheric turbulence, etc.) and when this velocity is reached instability occurs suddenly. For example, the state of divergence occurs suddenly, in a similar manner to the case of static loading of thin elastic shells. In general the situation here is very similar to that in the theory of the stability of thin shells.

The likelihood of the occurrence of these phenomena must be assessed for each particular system and for each particular range of Mach numbers. It should be borne in mind that due to viscosity or development and detachment of the boundary layer etc., the effects listed above are less pronounced, and under certain conditions aerodynamic non-linearity once more becomes "soft" (Fig. 128).

Two very simple examples will be described as an illustration. Consider the problem of the torsion of a thin wing, rectangular in plan, and with zero angle of attack. As a simplification, we shall assume that the wing is flat and that the displacement of the line of aerodynamic focal points relative to the stiffness axis is achieved by the presence of a leading edge which is not part of the load carrying part of the wing (see Fig. 127). We also assume that there is no possibility of flutter and that the change in entropy in transition through the shock wave can be ignored. We write the equation of torsion of the wing in the form

---

[1] Bautin, N. N. *The Behavior of Dynamic Systems in the Neighborhood of the Boundaries of the Region of Stability* (Povedenie dinamicheskikh sistem vblizi granits oblasti ustoichivosti). Gostekhizdat, 1949.

$$GJ_d \frac{d^2\theta}{dz^2} + \frac{p_\infty (b + x_0) x_0}{2} \left[\left(1 + \frac{\varkappa - 1}{2} \theta M \right)^{\frac{2\varkappa}{\varkappa - 1}} - \left(1 - \frac{\varkappa - 1}{2} \theta M \right)^{\frac{2\varkappa}{\varkappa - 1}} \right] = 0, \quad (4.112)$$

where θ is the angle of twist, GJ_d is the torsional stiffness. It can easily be shown that the critical value of the Mach number as found from the linearized equation

$$GJ_d \frac{d^2\theta}{dz^2} + \varkappa p_\infty (b + x_0) x_0 M \theta = 0$$

and the boundary conditions

$$\theta(0) = \frac{d\theta(l)}{dz} = 0,$$

is

$$M_* = \frac{\pi^2}{4l^2} \frac{GJ_d}{\varkappa p_\infty (b + x_0) x_0}.$$

We shall try to find a solution to the non-linear equation (4.112) in the form of a series in eigenfunctions of the corresponding linear problem

$$\theta(z) = \sum_{j=1}^{\infty} f_j \sin \frac{(2j - 1) \pi z}{2l}. \quad (4.113)$$

The results for a particular case with one term of Series (4.113) are plotted in Fig. 129. The branches corresponding to $d\theta/dM < 0$ are evidently unstable. When the critical value of $M_* = 9.15$ is reached, a sudden twist occurs up to a value of θ_k which lies within the region of separation angles of attack (see Fig. 128) and which cannot therefore be found from Equation (4.112).

In order to estimate the effect of initial inaccuracies, suppose that the wing has an initial twist given by

$$\theta = \theta_0 \sin \frac{\pi z}{2l}.$$

The corresponding equation is obtained from (4.112) if we replace θ in the square brackets by $\theta_0 + \theta$. The results are plotted in Fig. 129. It should be noted that there is a considerable reduction in the critical Mach number M_* due to initial effects; when $\theta_0 \doteq 0.01$ rad. the critical Mach number is $M_* = 8.2$, i.e. 10.5% lower than the upper critical value. When $\theta_0 = 0.02$ rad. the reduction is approximately 21%.

This example is characterized by the absence of geometric non-linearity (or more precisely, this non-linearity is negligibly small). We have the same state of affairs for other elements of the cantilever type in a flow of gas, for which the boundary of the stability region is "dangerous". For real structures of this sort instability can occur at a velocity less than the critical velocity given by the linearized theory for an idealized structure. On the other hand, for a panel the tangential deformations of which are resisted by the interaction with stiffening members, it is typical for geometric non-linearity to predominate.

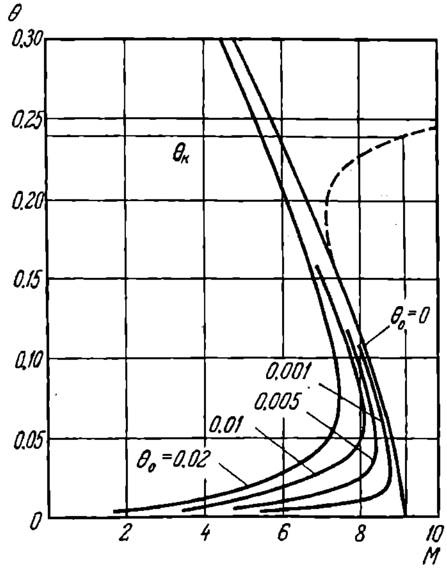


Fig. 129

As a very simple example consider the model provided by a plate lying on an elastic foundation and exposed to a flow of gas (Fig. 130). The question of the admissibility of applying the results to a plate of finite dimensions will be left open. The equation of disturbed motion of the plate is given by

$$D\,\nabla^2\nabla^2 w - N\frac{\partial^2 w}{\partial x^2} + k\,w + \varrho_0 h\frac{\partial^2 w}{\partial t^2} + \varrho_0 h\,\varepsilon\frac{\partial w}{\partial t} +$$

$$+ p_\infty\left[1 + \frac{\varkappa - 1}{2c_\infty}\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right)\right]^{\frac{2\varkappa}{\varkappa-1}} - p_\infty = 0, \tag{4.114}$$

where k is the stiffness of the foundation, N_x is the axial force in the plate, where

$$N_x = N_0 + \frac{c}{2\lambda_x}\int\limits_0^{\lambda_x}\left\{\sqrt{\left[1 + \left(\frac{\partial w}{\partial x}\right)^2\right]} - 1\right\}dx, \tag{4.115}$$

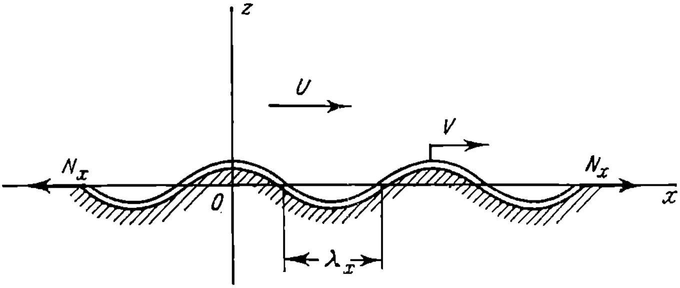


Fig. 130

λ_x is a characteristic length (which from now on will be taken as the half-wave length, and c is a stiffness coefficient characterizing resistance to decrease in length of the plate. This coefficient can vary within the limits

$$0 \leqq c \leqq \frac{E\,h}{1 - \mu^2}$$

and enables us to take into account the effect of geometric non-linearity on the behavior of the plate.

From consideration of the linearized problem we easily find that the critical velocity is

$$U_* = V(\lambda_x, \lambda_y)\left[1 + \frac{\varrho_\infty}{\varrho_0}\,\frac{\pi\,c_\infty\,\delta_0(\lambda_x, \lambda_y)}{h\,\Omega_0(\lambda_x, \lambda_y)}\right], \tag{4.116}$$

where $V(\lambda_x, \lambda_y)$ and $\Omega_0(\lambda_x, \lambda_y)$ is the phase velocity of propagation of elastic waves and the natural frequencies of the plate, both of which depend on the half-wave lengths λ_x and λ_y as parameters, $\delta_0(\lambda_x, \lambda_y)$ is the *in-vacuo* damping decrement of natural oscillations of the plate. There exist pairs of values of λ_x and λ_y which correspond to the minimum of Expression (4.116).

We note that the linearized equation has particular solutions of the form

$$w(x, y, t) = A \sin \frac{\pi y}{\lambda_y} \sin \left(\Omega t - \frac{\pi x}{\lambda_x}\right). \tag{4.117}$$

This solution is still exact if geometrical non-linearity alone is taken into account and the aerodynamic forces are linearized. We shall limit ourselves to non-linear terms of the third order. Expanding the square root in Formula (4.115) into a series, retaining terms of an order not higher than the second, and substituting the result into Equation (4.114), we obtain

$$D \nabla^2 \nabla^2 w - \left[N_0 + \frac{c}{2\lambda_x} \int\limits_0^{\lambda_x} \left(\frac{\partial w}{\partial x}\right)^2 dx\right] \frac{\partial^2 w}{\partial x^2} +$$

$$+ k w + \varrho_0 h \frac{\partial^2 w}{\partial t^2} + \varrho_0 h \varepsilon \frac{\partial w}{\partial t} + \frac{\varkappa p_\infty}{c_\infty} \left(\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x}\right) = 0.$$

Since from (4.117)

$$\int\limits_0^{\lambda_x} \left(\frac{\partial w}{\partial x}\right)^2 dx = \frac{\pi^2 A^2}{2\lambda_x},$$

we obtain the following equations for the amplitude A and frequency Ω:

$$\left.\begin{aligned} \left[D\left(\frac{\pi^2}{\lambda_x^2} + \frac{\pi^2}{\lambda_y^2}\right)^2 + \left(N_0 + \frac{\pi^2 c A^2}{4\lambda_x^2}\right)\frac{\pi^2}{\lambda_x^2} + k - \varrho_0 h \Omega^2\right] A &= 0, \\ \left[\varrho_0 h \varepsilon \Omega + \frac{\varkappa p_\infty}{c_\infty}\left(\Omega - \frac{\pi U}{\lambda_x}\right)\right] A &= 0. \end{aligned}\right\} \tag{4.118}$$

Making the stipulation that $A \neq 0$, we find from the second equation that

$$\frac{\Omega}{\Omega_0} = \frac{U}{U_*},$$

where $\Omega_0(\lambda_x, \lambda_y)$ is the *in-vacuo* natural frequency of the plate and $U_*(\lambda_x, \lambda_y)$ is the critical velocity (4.116). Thus from the first equation,

$$A = \frac{h}{S}\sqrt{\left[\left(\frac{U}{U_*}\right)^2 - 1\right]}, \tag{4.119}$$

where

$$S = \frac{\pi}{2\lambda_x h\Omega_0}\sqrt{\frac{c}{\varrho_0}}.$$

Calculations based on Formula (4.119) show that if the stiffness coefficient c has a maximum (this signifies that the length of the projection of the deformed plate in the direction of the axis Ox remains constant), then the amplitude A is of the order of h even at velocities more than twice the critical velocity U_*.

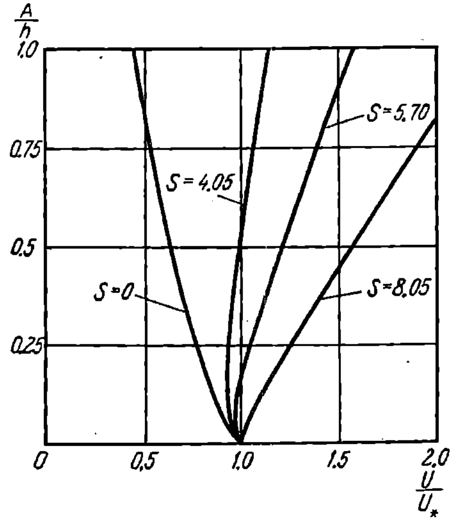


Fig. 131

If aerodynamic non-linearity is taken into account, the Solution (4.117) is no longer exact. However, an approximate solution can be found by discarding terms containing trigonometric functions of arc lengths which are multiple with respect to $\Omega\, t - \Pi\, x/\lambda_x$. For example,

$$\left(\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x}\right)^3 \approx \frac{3}{4}A^3\left(\Omega - \frac{\pi U}{\lambda_x}\right)^3 \cos\left(\Omega\, t - \frac{\pi x}{\lambda_x}\right) + \cdots$$

The first equation of (4.118) remains unaltered, but the second now becomes

$$\left[\varrho_0 h \varepsilon \Omega + \frac{\varkappa p_\infty}{c_\infty}\left(\Omega - \frac{\pi U}{\lambda_x}\right) + \frac{3}{4}A^2\left(\Omega - \frac{\pi U}{\lambda_x}\right)^3 \times \right.$$
$$\left. \times \frac{\varkappa p_\infty(\varkappa + 1)}{12 c_\infty^3}\right] A = 0.$$

The solutions to the equations for certain specified data are shown in Fig. 131. The solution depends on the stiffness parameter S. If $S = 8.05$ (with the data chosen for this example this corresponds to the upper limit for c), we have "soft" excitation of travelling waves. As S decreases transition occurs to "hard" excitation.

4.13. DERIVATION OF THE EQUATIONS OF NON-LINEAR FLUTTER OF A SHALLOW SHELL AT HIGH SUPERSONIC VELOCITIES

Consider first the equilibrium of an elastic shell supported on a contour rectangular in plan and exposed on one side to a supersonic flow of gas with undisturbed velocity U in a direction parallel to the axis Ox_0 (Fig. 132). We shall assume that the shell is sufficiently shallow to enable us to identify approximately the metric of its middle surface with the euclidean metric. We shall employ the usual hypotheses of the non-linear theory of elastic shells[1], assuming that the Kirchhoff-Love hypothesis is valid and that the deflections of

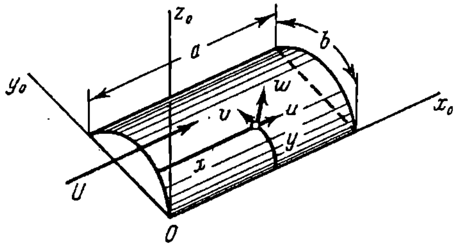


Fig. 132

---

[1] VOL'MIR, A. S. *Flexible Plates and Shells* (Gibkie plastinki i obolochki). Gostekhizdat, 1956; MUSHTARI, Kh. M., GALIMOV, K. Z. *Non-linear Theory of Elastic Shells* (Nelineinaia teoriia uprugikh obolochek). Tatknigoizdat, 1957. English translation published for NSF and NASA, 1961.

the shell are comparable with its thickness h, but small compared with its sides a and b. We shall suppose that the distribution of temperature T for the shell is specified and that it can be expressed as the sum of two terms, as follows:

$$T = T_0(x, y) + z\,\Theta(x, y). \tag{4.120}$$

Here $T_0(x, y)$ is the temperature on the middle surface and $\Theta(x, y)$ is the temperature gradient in a normal direction[1], x and y are curvilinear coordinates measured along the lines of curvature of the undeformed middle surface, z is a coordinate measured along the external normal to the middle surface of the shell (i.e. towards the side exposed to the flow). Finally, we assume that the mechanical properties of the material of the shell depend on the temperature T, but that their variation between different points on the shell is only small. This assumption is justified if the non-uniformity of the temperature field within the limits of the shell is not large and that it can be removed by a simple generalization of the appropriate equations.

The components of the strain tensor at every point on the shell, ε_x, ε_y and ε_{xy} are related to the components of the stress tensor σ_x, σ_y and σ_{xy} as follows:

$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E}(\sigma_x - \mu\,\sigma_y) + \alpha\,T, \\ \varepsilon_y &= \frac{1}{E}(\sigma_y - \mu\,\sigma_x) + \alpha\,T, \\ \varepsilon_{xy} &= \frac{1+\mu}{E}\sigma_{xy}, \end{aligned}\right\} \tag{4.121}$$

where α is the coefficient of thermal expansion. The inverse relations are

$$\left.\begin{aligned} \sigma_x &= \frac{E}{1-\mu^2}[\varepsilon_x + \mu\,\varepsilon_y - (1+\mu)\,\alpha\,T], \\ \sigma_y &= \frac{E}{1-\mu^2}[\varepsilon_y + \mu\,\varepsilon_x - (1+\mu)\,\alpha\,T], \\ \sigma_{xy} &= \frac{E}{1+\mu}\varepsilon_{xy}. \end{aligned}\right\} \tag{4.122}$$

---

[1] The determination of the functions T_0 and Θ is described in the paper: Bolotin, V. V. Equations of non-steady temperature fields in thin shells in the presence of heat sources. *PMM*, **24**, No. 2 (1960).

In addition to the components ε_x, ε_y and ε_{xy}, we introduce the components $\bar{\varepsilon}_x$, $\bar{\varepsilon}_y$ and $\bar{\varepsilon}_{xy}$ which are mean values over the thickness h of the shell. They can be looked upon as the strain components of the middle surface:

$$\left.\begin{aligned}\bar{\varepsilon}_x &= \frac{1}{h}\int_{-h/2}^{h/2} \varepsilon_x\, dz = \frac{1}{Eh}(N_x - \mu N_y) + \alpha T_0,\\ \bar{\varepsilon}_y &= \frac{1}{h}\int_{-h/2}^{h/2} \varepsilon_y\, dz = \frac{1}{Eh}(N_y - \mu N_x) + \alpha T_0,\\ \bar{\varepsilon}_{xy} &= \frac{1}{h}\int_{-h/2}^{h/2} \varepsilon_{xy}\, dz = \frac{1+\mu}{Eh} N_{xy}.\end{aligned}\right\} \tag{4.123}$$

Here N_x, N_y and N_{xy} are forces per unit length of shell section, i.e.

$$N_x = \int_{-h/2}^{h/2} \sigma_x\, dz, \qquad N_y = \int_{-h/2}^{h/2} \sigma_y\, dz, \qquad N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy}\, dz.$$

On the other hand, for the components of the strain tensor $\bar{\varepsilon}_x$, $\bar{\varepsilon}_y$ and $\bar{\varepsilon}_{xy}$ we have the purely geometrical relations

$$\left.\begin{aligned}\bar{\varepsilon}_x &= \frac{\partial u}{\partial x} - k_x w + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2,\\ \bar{\varepsilon}_y &= \frac{\partial v}{\partial y} - k_y w + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,\\ \bar{\varepsilon}_{xy} &= \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}.\end{aligned}\right\} \tag{4.124}$$

Here $u(x, y)$, $v(x, y)$ and $w(x, y)$ are the tangential and normal displacements of points on the middle surface, k_x and k_y are the principal curvatures of the undeformed middle surface, taken as positive if the center of curvature lies on the external side of the shell (the side exposed to the flow). Eliminating the deflections u and v from Formulae (4.124), we obtain the condition of strain compatibility

$$\frac{\partial^2 \bar{\varepsilon}_x}{\partial y^2} + \frac{\partial^2 \bar{\varepsilon}_y}{\partial x^2} - 2\frac{\partial^2 \bar{\varepsilon}_{xy}}{\partial x\, \partial y} = \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - k_x \frac{\partial^2 w}{\partial y^2} - k_y \frac{\partial^2 w}{\partial x^2}. \tag{4.125}$$

Substitution of Formulae (4.123) into Condition (4.125) gives

$$\frac{1}{Eh}\left(\frac{\partial^2 N_x}{\partial y^2} - 2\frac{\partial^2 N_{xy}}{\partial x\,\partial y} + \frac{\partial^2 N_y}{\partial x^2}\right) -$$

$$-\frac{\mu}{Eh}\left(\frac{\partial^2 N_x}{\partial y^2} + 2\frac{\partial^2 N_{xy}}{\partial x\,\partial y} + \frac{\partial^2 N_y}{\partial y^2}\right) + \alpha\,\nabla^2 T_0 =$$

$$= \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - k_x\frac{\partial^2 w}{\partial y^2} - k_y\frac{\partial^2 w}{\partial x^2}.$$

As usual, we express the forces in the middle surface in terms of a function $\Phi(x, y)$:

$$N_x = \frac{\partial^2 \Phi}{\partial y^2}, \qquad N_y = \frac{\partial^2 \Phi}{\partial x^2}, \qquad N_{xy} = -\frac{\partial^2 \Phi}{\partial x\,\partial y}. \qquad (4.126)$$

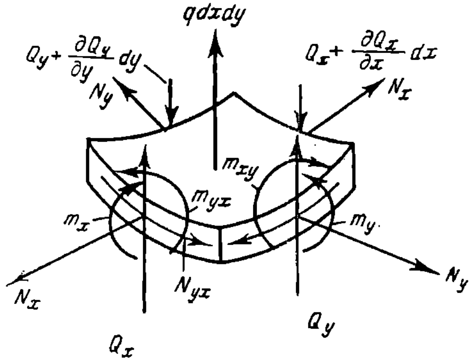

$q\,dx\,dy$
$Q_y+\frac{\partial Q_y}{\partial y}dy$
N_y
$Q_x+\frac{\partial Q_x}{\partial x}dx$
N_x
m_{xy}
m_{yx}
m_x
m_y
N_x
N_{yx}
N_y
Q_y
Q_x

Fig. 133

Then the equilibrium equations

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \qquad \frac{\partial N_{yx}}{\partial x} + \frac{\partial N_y}{\partial y} = 0, \qquad N_{yx} = N_{xy}$$

will automatically be satisfied and we obtain one equation relating the deflection function $w(x, y)$ and the function $\Phi(x, y)$:

$$\frac{1}{Eh}\nabla^2\nabla^2\Phi + \alpha\,\nabla^2 T_0 =$$

$$= \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - k_x\frac{\partial^2 w}{\partial y^2} - k_y\frac{\partial^2 w}{\partial x^2}. \qquad (4.127)$$

In order to obtain a second equation we must consider the equilibrium equations which relate the shear forces Q_x and Q_y to the normal loading q (Fig. 133):

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x\left(k_x + \frac{\partial^2 w}{\partial x^2}\right) + \\ + N_y\left(k_y + \frac{\partial^2 w}{\partial y^2}\right) + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y} + q = 0,$$

as well as the equations which relate the shear forces to the bending moments m_x and m_y and to the twisting moments $m_{xy} = m_{yx}$:

$$Q_x = \frac{\partial m_x}{\partial x} + \frac{\partial m_{xy}}{\partial y}, \qquad Q_y = \frac{\partial m_{yx}}{\partial x} + \frac{\partial m_y}{\partial y}.$$

After eliminating the shear forces, we obtain

$$\frac{\partial^2 m_x}{\partial x^2} + 2\frac{\partial^2 m_{xy}}{\partial x\,\partial y} + \frac{\partial^2 m_y}{\partial y^2} + \\ + N_x\left(k_x + \frac{\partial^2 w}{\partial x^2}\right) + N_y\left(k_y + \frac{\partial^2 w}{\partial y^2}\right) + 2N_{xy}\frac{\partial^2 w}{\partial x\,\partial y} + q = 0. \tag{4.128}$$

We now express the moments m_x, m_y and m_{xy} in terms of the deflection function $w(x, y)$. Note first that

$$m_x = \int_{-h/2}^{h/2} \sigma_x z\,dz, \qquad m_y = \int_{-h/2}^{h/2} \sigma_y z\,dz, \qquad m_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} z\,dz. \tag{4.129}$$

We substitute here Expressions (4.122). Noting that by virtue of the Kirchhoff–Love hypothesis and Formula (4.121),

$$\varepsilon_x = \bar{\varepsilon}_x - z\frac{\partial^2 w}{\partial x^2} - \alpha z \Theta,$$

$$\varepsilon_y = \bar{\varepsilon}_y - z\frac{\partial^2 w}{\partial y^2} - \alpha z \Theta,$$

$$\varepsilon_{xy} = \bar{\varepsilon}_{xy} - z\frac{\partial^2 w}{\partial x\,\partial y},$$

and making use of Relation (4.123), we find by substitution into Formulae (4.129), that

$$\left.\begin{aligned} m_x &= -D\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2} + \alpha(1+\mu)\Theta\right], \\ m_y &= -D\left[\frac{\partial^2 w}{\partial y^2} + \mu\frac{\partial^2 w}{\partial x^2} + \alpha(1+\mu)\Theta\right], \\ m_{xy} &= -D(1-\mu)\frac{\partial^2 w}{\partial x\,\partial y} \end{aligned}\right\} \tag{4.130}$$

(here D is the cylindrical stiffness). We now substitute Expressions (4.130) and (4.126) into Equation (4.128):

$$D \nabla^2 \nabla^2 w + \alpha(1 + \mu) D \nabla^2 \Theta =$$
$$= \frac{\partial^2 \Phi}{\partial y^2}\left(k_x + \frac{\partial^2 w}{\partial x^2}\right) + \frac{\partial^2 \Phi}{\partial x^2}\left(k_y + \frac{\partial^2 w}{\partial y^2}\right) - 2\frac{\partial^2 \Phi}{\partial x\, \partial y}\, \frac{\partial^2 w}{\partial x\, \partial y} + q. \tag{4.131}$$

We obtain the oscillation equations by introducing into Equations (4.131) and (4.127) terms which take into account inertial and dissipative forces and the pressure of the gas:

$$q = -\varrho_0 h \frac{\partial^2 w}{\partial t^2} - \varrho_0 h \varepsilon \frac{\partial w}{\partial t} + p_0 - p$$

(where p_0 is the internal pressure). The effect of tangential forces will be ignored. This can be done in practically every case, since the natural frequencies of tangential oscillations of a thin shallow shell are always sufficiently high compared, at any rate, with the lower natural frequencies of normal oscillations. As regards the gas pressure, we express it according to Formula (4.78), retaining three terms of the expansion[1]. As a result we arrive at the system of equations

$$D \nabla^2 \nabla^2 w + \alpha(1 + \mu) D \nabla^2 \Theta + \varrho_0 h \frac{\partial^2 w}{\partial t^2} + \varrho_0 h \varepsilon \frac{\partial w}{\partial t} =$$
$$= \left(k_x + \frac{\partial^2 w}{\partial x^2}\right)\frac{\partial^2 \Phi}{\partial y^2} + \left(k_y + \frac{\partial^2 w}{\partial y^2}\right)\frac{\partial^2 \Phi}{\partial x^2} - 2\frac{\partial^2 w}{\partial x\, \partial y}\, \frac{\partial^2 \Phi}{\partial x\, \partial y} +$$
$$+ p - p_\infty - \varkappa p_\infty\left(\frac{\partial w}{\partial t} + U\frac{\partial w_1}{\partial x_0}\right) - \frac{\varkappa(\varkappa + 1)p_\infty}{4}\left(\frac{\partial w}{\partial t} + U\frac{\partial w_1}{\partial x_0}\right)^2 -$$
$$- \frac{\varkappa(\varkappa + 1)p_\infty}{12}\left(\frac{\partial w}{\partial t} + U\frac{\partial w_1}{\partial x_0}\right)^3 = 0,$$

$$\frac{1}{Eh}\nabla^2 \nabla^2 \Phi + \alpha \nabla^2 T_0 =$$
$$= \left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\, \frac{\partial^2 w}{\partial y^2} - k_x \frac{\partial^2 w}{\partial y^2} - k_y \frac{\partial^2 w}{\partial x^2}, \tag{4.132}$$

where $w_1 = w_0 + w$, $w_0 = w_0(x_0, y)$ is the equation of the undeformed middle surface.

---

[1] Equations (4.127) and (4.131) also correspond to taking into account nonlinear terms of an order not higher than the third with respect to w.

It should be noted that if the mean temperature $T_0(x, y)$ and the gradient $\Theta(x, y)$ are constant over the whole of the shell, or are linear functions of the coordinates of the middle surface x, y, then the temperature terms in Equations (4.132) vanish. However, the temperature terms in the boundary conditions remain. The question of boundary conditions will be discussed later, when the question of the method of solution is discussed.

4.14. APPROXIMATE METHOD OF SOLUTION OF THE EQUATIONS

The complexity of the system of Equations (4.132) prompts us to try to find an approximate solution. One approximate method will now be given[1]. Its particular advantage lies in the fact that it enables us to take into account non-linear boundary conditions for tangential deformation.

We express the function $w(x, y, t)$ in the form of an expansion in the functions $w_j(x, y)$ which satisfy all the boundary conditions for the normal deflection

$$w(x, y, t) = \sum_{j=1}^{n} f_j(t)\, w_j(x, y). \tag{4.133}$$

The boundary conditions for the normal deflection are formulated in the usual manner. If the edge $x = 0$ is fully fixed, then

$$w\Big|_{x=0} = \frac{\partial w}{\partial x}\Big|_{x=0} = 0. \tag{4.134}$$

If this edge is simply supported on a contour which is absolutely rigid in a normal direction, then the bending moment m_x, as well as the deflection, must be zero. Taking into account Relation (4.130), we obtain

$$w\Big|_{x=0} = \frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2} + \alpha(1 + \mu)\,\Theta\Big|_{x=0} = 0.$$

The most general case is when the edge is supported by a structural member (for instance a stringer or a frame) which possesses a bending stiffness B in a normal direction and a torsional stiffness C. The conditions on the edge $x = 0$ for this case become

---

[1] BOLOTIN, V. V. Non-linear flutter of plates and shells, *Inzh. sborn.*, **28** (1960). In a less general form this method was used in the study: BOLOTIN, V. V., GAVRILOV, Iu. V., MAKAROV, B. P., SHVEIKO, Iu. Iu. Non-linear problems of the stability of flat panels at high supersonic velocities. *Izv. Akad. Nauk. USSR, OTN, "Mekhanika i mashinostroenie"*, No. 3 (1959).

$$B\frac{\partial^4 w}{\partial y^4}\bigg|_{x=0} = \left(Q_x + \frac{\partial m_{xy}}{\partial y}\right)\bigg|_{x=0},$$

$$C\frac{\partial^3 x}{\partial x\,\partial y^2}\bigg|_{x=0} = m_x\bigg|_{x=0}.$$

Substitution of the expressions for m_x, m_{xy} and Q_x yields

$$B\frac{\partial^4 w}{\partial x^4}\bigg|_{x=0} = -D\left[\frac{\partial^3 w}{\partial x^3} + (2-\mu)\frac{\partial^3 w}{\partial x\,\partial y^2} + \alpha(1+\mu)\frac{\partial \Theta}{\partial x}\right]_{x=0},$$

$$C\frac{\partial^3 w}{\partial x\,\partial y^2}\bigg|_{x=0} = -D\left[\frac{\partial^2 w}{\partial x^2} + \mu\frac{\partial^2 w}{\partial y^2} + \alpha(1+\mu)\,\Theta\right]_{x=0}.$$

If two neighboring panels are supported on the same member, the boundary conditions must be generalized. It should be noted that if the temperature gradient Θ is non-zero the boundary conditions, apart from (4.134), are non-homogeneous. This makes it difficult to select functions $w_j(x, y)$ which satisfy the necessary conditions.

Suppose that a system of functions $w_j(x, y)$ has been found. Then, substituting the series (4.133) into the right-hand side of the second equation of (4.132), we obtain an equation which is linear with respect to the stress function $\Phi(x, y, t)$ (here time is looked upon as a parameter). Its general solution, as usual, contains arbitrary functions, by means of which we can satisfy the boundary conditions for the function $\Phi(x, y, t)$. However, the conditions which ensure the combined action of the shell and its stiffening structure are usually so complicated than an exact formulation of the boundary conditions for $\Phi(x, y, t)$ ceases to have any practical meaning. It is sufficient to formulate these conditions "on the average", and to characterize the interaction of the shell and the adjacent structure by certain mean parameters.[1]

We express the stress function in the approximate form

$$\Phi(x, y, t) = \Phi_1(x, y, t) + \frac{1}{2}(\bar{N}_x y^2 + \bar{N}_y x^2 - 2\bar{N}_{xy} x y). \quad (4.135)$$

Here $\Phi_1(x, y, t)$ is some particular solution of the second of Equations (4.132), $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$ are parameters chosen in such a way that the conditions for the function $\Phi(x, y, t)$ are satisfied "on the average". It is very convenient to take an expression for the function $\Phi_1(x, y, t)$

---

[1] For non-linear static problems of plates and shells, this procedure was first suggested by P. F. Papkovich.

which satisfies the conditions

$$\left.\begin{aligned}
&\int_0^b \left.\frac{\partial^2\Phi_1}{\partial y^2}\right|_{x=0} dy = \int_0^b \left.\frac{\partial^2\Phi_1}{\partial y^2}\right|_{x=a} dy = 0,\\
&\int_0^a \left.\frac{\partial^2\Phi_1}{\partial x^2}\right|_{y=0} dx = \int_0^a \left.\frac{\partial^2\Phi_1}{\partial x^2}\right|_{y=b} dx = 0,\\
&\int_0^b \left.\frac{\partial^2\Phi_1}{\partial x\,\partial y}\right|_{x=0} dy = \int_0^b \left.\frac{\partial^2\Phi_1}{\partial x\,\partial y}\right|_{x=a} dy =\\
&\quad = \int_0^a \left.\frac{\partial^2\Phi_1}{\partial x\,\partial y}\right|_{y=0} dx = \int_0^a \left.\frac{\partial^2\Phi_1}{\partial x\cdot\partial y}\right|_{y=b} dx = 0.
\end{aligned}\right\} \tag{4.136}$$

Then the parameters $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$ assume the meaning of average forces along the edges:

$$\left.\begin{aligned}
&\frac{1}{b}\int_0^b \left.\frac{\partial^2\Phi}{\partial y^2}\right|_{x=0} dy = \bar{N}_x,\\
&\frac{1}{a}\int_0^a \left.\frac{\partial^2\Phi}{\partial x^2}\right|_{y=0} dx = \bar{N}_y,\\
&\frac{1}{a}\int_0^a \left.\frac{\partial^2\Phi}{\partial x\,\partial y}\right|_{y=0} dx = \bar{N}_{xy}.
\end{aligned}\right\} \tag{4.137}$$

From now on we shall assume that Conditions (4.136) are satisfied and consequently, that Relations (4.137) hold.

In addition to the mean forces, we introduce mean relative displacement of the edges and a relative shear

$$\left.\begin{aligned}
\bar{\Delta}_x &= -\frac{1}{ab}\int_0^a\int_0^b \frac{\partial u}{\partial x}\,dx\,dy,\\
\bar{\Delta}_y &= -\frac{1}{ab}\int_0^a\int_0^b \frac{\partial v}{\partial y}\,dx\,dy,\\
\bar{\Delta}_{xy} &= -\frac{1}{2ab}\int_0^a\int_0^b \left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) dx\,dy.
\end{aligned}\right\} \tag{4.138}$$

The boundary conditions for the function $\Phi(x, y, t)$ can now be expressed in the form

$$\bar{N}_x = c_x \bar{\Delta}_x, \qquad \bar{N}_y = c_y \bar{\Delta}_y, \qquad \bar{N}_{xy} = c_{xy} \bar{\Delta}_{xy}, \tag{4.139}$$

where c_x, c_y and c_{xy} are coefficients which characterize the resistance of the stiffening structure to relative displacement of the edges and to angular changes at the contour at which the shell is supported. These conditions are sufficient to find the three parameters $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$.

In order to apply Conditions (4.139), we must express the derivatives of the displacements u and v in Formulae (4.138) in terms of the functions Φ and w. We find from (4.123) and (4.124) that

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{1}{E h}(N_x - \mu N_y) + \alpha T_0 + k_x w - \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \\ \frac{\partial v}{\partial y} &= \frac{1}{E h}(N_y - \mu N_x) + \alpha T_0 + k_y w - \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) &= \frac{1+\mu}{E h} N_{xy} - \frac{1}{2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}. \end{aligned}\right\} \tag{4.140}$$

After the parameters $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$ have been found, Expression (4.135) and Series (4.133) are substituted into the first of Equations (4.132). Writing this equation in the shortened form $\mathfrak{L}(w) = 0$, we make the requirement that it be satisfied approximately in the sense of Galerkin's method:

$$\int_0^a \int_0^b \mathfrak{L}(w)\, w_j(x, y)\, dx\, dy = 0 \qquad (j = 1, 2, \ldots, n).$$

From this we obtain a system of ordinary differential equations in the coefficients of Series (4.133). If for the functions $w_j(x, y)$ we take the modes of small natural oscillations of a panel with natural frequencies Ω_j, then the system of equations assumes the form

$$\frac{d^2 f_j}{d t^2} + \varepsilon \frac{d f_j}{d t} + \Omega_j^2 f_j + F_j(f_1, f_2, \ldots, f_n, M) = 0 \tag{4.141}$$

$$(j = 1, 2, \ldots, n).$$

Here F_j are, in general, non-linear functions of $f_1, f_2, \ldots, f_n$ which depend on the parameters of the panel and the Mach number M. The system of equations so derived allows a simpler analytical investigation than the initial system of Equations (4.132).

It now becomes expedient to split the solution of the problem into two stages. The first stage is the solution of the problem of thermal buckling of the panel and the second is the solution of the problem of flutter of the buckled panel. Let us start with the first stage. Let $w = w_T(x, y)$, $\Phi = \Phi_T(x, y)$. Then instead of (4.132) we obtain

$$\left.\begin{aligned}
&D\,\nabla^2\nabla^2 w_T + \alpha(1+\mu)\,D\,\nabla^2\Theta = \\
&\quad = \left(k_x + \frac{\partial^2 w_T}{\partial x^2}\right)\frac{\partial^2\Phi_T}{\partial y^2} + \left(k_y + \frac{\partial^2 w_T}{\partial y^2}\right)\frac{\partial^2\Phi_T}{\partial x^2} - \\
&\quad - 2\frac{\partial^2 w_T}{\partial x\,\partial y}\frac{\partial^2\Phi_T}{\partial x\,\partial y} + p_0 - p_\infty - \varkappa p_\infty M\frac{\partial(w_0 + w_T)}{\partial x_0} - \\
&\qquad - \frac{\varkappa(\varkappa+1)\,p_\infty M^2}{4}\left[\frac{\partial(w_0 + w_T)}{\partial x_0}\right]^2 - \\
&\qquad\qquad - \frac{\varkappa(\varkappa+1)\,p_\infty M^3}{12}\left[\frac{\partial(w_0 + w_T)}{\partial x_0}\right]^3, \\
&\frac{1}{Eh}\nabla^2\nabla^2\Phi_T + \alpha\,\nabla^2 T_0 = \\
&= \left(\frac{\partial^2 w_T}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w_T}{\partial x^2}\frac{\partial^2 w_T}{\partial y^2} - k_x\frac{\partial^2 w_T}{\partial y^2} - k_y\frac{\partial^2 w_T}{\partial x^2}.
\end{aligned}\right\} \tag{4.142}$$

If the temperature field $T(x, y)$ is known, then by expressing the normal deflection in the form of a series

$$w_T(x, y) = \sum_{j=1}^{n} f_j^T\, w_j(x, y) \tag{4.143}$$

and adopting the method described above, we can find approximate expressions for the function $\Phi_T(x, y)$ and the deflections $w_T(x, y)$. Note that the problem of thermal buckling is simplified if it is assumed that the temperature is constant through the thickness of the panel and that the temperature $T_0(x, y)$ is a linear function of the coordinates. In this case the temperature terms in Equations (4.142) vanish and only remain in the boundary conditions for Φ_T.

Consider once again Equations (4.132). If we set $w = w_T + w_F$, $\Phi = \Phi_T + \Phi_F$ in the first of these and bear in mind that w_T and Φ_T

satisfy Equations (4.142), we obtain

$$\begin{aligned}D\,\nabla^2\nabla^2 w_F &= \frac{\partial^2\Phi_T}{\partial y^2}\frac{\partial^2 w_F}{\partial x^2} + \frac{\partial^2\Phi_T}{\partial x^2}\frac{\partial^2 w_F}{\partial y^2} - \\ &\quad - 2\frac{\partial^2\Phi_T}{\partial x\,\partial y}\frac{\partial^2 w_F}{\partial x\,\partial y} + \frac{\partial^2\Phi_F}{\partial y^2}\left[k_x + \frac{\partial^2(w_T + w_F)}{\partial x^2}\right] + \\ &+ \frac{\partial^2\Phi_F}{\partial x^2}\left[k_y + \frac{\partial^2(w_T + w_F)}{\partial y^2}\right] - 2\frac{\partial^2\Phi_F}{\partial x\,\partial y}\frac{\partial^2(w_T + w_F)}{\partial x\,\partial y} + q,\end{aligned} \tag{4.144}$$

$$\begin{aligned}q &= -\varrho_0 h\frac{\partial^2 w_F}{\partial t^2} - \left(\varrho_0 h\varepsilon + \frac{\varkappa p_\infty}{c_\infty}\right)\frac{\partial w_F}{\partial t} - \varkappa p_\infty M\left(\frac{\partial w_F}{\partial x_0}\right) - \\ &\quad - \frac{\varkappa(\varkappa + 1)p_\infty M^2}{4}\left[2\frac{\partial w_F}{\partial x_0}\frac{\partial(w_0 + w_T)}{\partial x_0} + \left(\frac{\partial w_F}{\partial x_0}\right)^2\right] - \\ &\quad - \frac{\varkappa(\varkappa + 1)p_\infty M^3}{12}\left\{3\frac{\partial w_F}{\partial x_0}\left[\frac{\partial(w_0 + w_T)}{\partial x_0}\right]^2 + \right. \\ &\qquad \left. + 3\left(\frac{\partial w_F}{\partial x_0}\right)^2\frac{\partial(w_0 + w_T)}{\partial x_0} + \left(\frac{\partial w_F}{\partial x_0}\right)^3\right\}\end{aligned} \tag{4.145}$$

(of the terms which take aerodynamic damping into account we have retained only the most significant linear term). The second equation becomes

$$\begin{aligned}\frac{1}{Eh}\nabla^2\nabla^2\Phi_F &= \left(\frac{\partial^2 w_F}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w_F}{\partial x^2}\frac{\partial^2 w_F}{\partial y^2} - \\ &- \left[\left(k_x + \frac{\partial^2 w_T}{\partial x^2}\right)\frac{\partial^2 w_F}{\partial y^2} + \left(k_y + \frac{\partial^2 w_T}{\partial y^2}\right)\frac{\partial^2 w_F}{\partial x^2} - 2\frac{\partial^2 w_T}{\partial x\,\partial y}\frac{\partial^2 w_F}{\partial x\,\partial y}\right].\end{aligned} \tag{4.146}$$

Now consider the conditions on the contour

$$\overline{N}_x^F = c_x\overline{\Delta}_x^F,\qquad \overline{N}_y^F = c_y\overline{\Delta}_y^F,\qquad \overline{N}_{xy}^F = c_{xy}\overline{\Delta}_{xy}^F. \tag{4.147}$$

The mean value of the forces and displacements are given by Formulae (4.137) and (4.138) by replacing N_x by N_x^F, N_y by N_y^F etc. Also, $u_F = u - u_T$, $v_F = v - v_T$. As a result, the boundary Conditions (4.147) are expressed in terms of $w_F(x, y, t)$ and $\Phi_F(x, y, t)$.

4.15. PANEL SUPPORTED OVER ITS ENTIRE CONTOUR

Suppose that the first part of the problem—the investigation of temperature buckling of a panel—has been solved, i.e. that $w_T(x, y)$ and $\Phi_T(x, y)$ are known. We now proceed to the second and more difficult part of the problem. If the temperature deflection is given in the form of Series (4.133), it is expedient to try to find the dynamic flutter deflection also in the form

$$w_F(x, y, t) = \sum_{j=1}^{n} f_j^F(t)\, w_j(x, y). \tag{4.148}$$

Substituting this series into the right-hand side of Equation (4.146), we find that its solution is of the type (4.135) and contains three constants $\bar{N}_x$, $\bar{N}_y$ and $\bar{N}_{xy}$, the values of which can be found from the conditions on the contour (4.147). Series (4.148) and the expression found for $\Phi_F(x, y, t)$ are then substituted into Equation (4.144), which is satisfied approximately in the sense of Galerkin's method. A system of ordinary differential equations of the type (4.141) must result from these transformations.

As an example, consider the case of a panel supported along its edges which assumes a buckled form defined by

$$w_T(x, y) = f_T \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}. \tag{4.149}$$

We express the dynamic deflection in the form of two terms of the series as follows:

$$w_F(x, y, t) = f_1(t) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + f_2(t) \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b}. \tag{4.150}$$

Judging by the results which the approximation (4.150) gives for the linear problem (Section 4.10), this approximation can also be used for a preliminary quantitative analysis of the non-linear problem. Substituting Expression (4.150) into the right-hand side of Equation (4.146), we obtain

$$\Phi_F(x, y, t) = \Phi_1(x, y, t) + \frac{1}{2}(\bar{N}_x y^2 + \bar{N}_y x^2 - 2\bar{N}_{xy}\, x y), \tag{4.151}$$

where

$$\begin{aligned}
\Phi_1(x, y, t) = \frac{E h \varphi^2}{4}\Bigg[&- (f_T + f_1) f_2 \cos\frac{\pi x}{a} + \\
&+ \frac{(2f_T + f_1) f_1}{8} \cos\frac{2\pi x}{a} + \frac{(f_T + f_1) f_2}{9} \cos\frac{3\pi x}{a} + \frac{f_2^2}{32} \cos\frac{4\pi x}{a} + \\
&+ \frac{9(f_T + f_1) f_2}{(1 + 4\varphi^2)^2} \cos\frac{\pi x}{a} \cos\frac{2\pi y}{b} - \frac{(f_T + f_1) f_2}{(9 + 4\varphi^2)^2} \cos\frac{3\pi x}{a} \cos\frac{2\pi y}{b} + \\
&+ \frac{(2f_T + f_1) f_1 + 4 f_2^2}{8\varphi^2} \cos\frac{2\pi y}{b} + \frac{4 b^2 (\varphi^2 k_x + k_y) f_1}{\pi^2 (1 + \varphi^2)^2} \sin\frac{\pi x}{a} \sin\frac{\pi y}{b} + \\
&+ \frac{4 b^2 (\varphi^2 k_x + 4 k_y) f_2}{\pi^2 (4 + \varphi^2)^2} \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b}\Bigg].
\end{aligned}$$

Here, as before, we have used the notation $\varphi = a/b$.

We now express N_x^F, N_y^F and N_{xy}^F in terms of Φ_F and substitute them together with the expressions for w_F into Formulae (4.138):

$$\Delta_x^F = -\frac{(\overline{N}_x - \mu\,\overline{N}_y)}{Eh} + \frac{4f_1(\varphi^2 k_x + k_y)(\varphi^2 - \mu)}{\pi^2(1+\varphi^2)^2} - \\ - \frac{4k_x f_1}{\pi^2} + \frac{\pi^2}{8a^2}[(2f_T + f_1)f_1 + 4f_2^2],$$

$$\Delta_y^F = -\frac{(\overline{N}_y - \mu\,\overline{N}_x)}{Eh} + \frac{4f_1(\varphi^2 k_x + k_y)(1 - \mu\,\varphi^2)}{\pi^2(1+\varphi^2)^2} - \\ - \frac{4k_y f_1}{\pi^2} + \frac{\pi^2}{8b^2}[(2f_T + f_1)f_1 + f_2^2].$$

The expression for the shear deflection Δ_{xy}^F will not be derived, since we shall ignore resistance of the stiffening members to distortion of the contour of the shell in plan and put $c_{xy} = 0$. Making use of boundary Conditions (4.147), we find that

$$\left.\begin{aligned} \overline{N}_x &= \frac{\pi^2 E h \beta_x}{8a^2(1 - \mu^2 \beta_x \beta_y)}\{(2f_T + f_1)f_1 + 4f_2^2 - B_x h f_1 + \\ &\qquad + \mu\,\beta_y\,\varphi^2[(2f_T + f_1)f_1 + f_2^2 + B_y h f_1]\}, \\ \overline{N}_y &= \frac{\pi^2 E h \beta_y}{8a^2(1 - \mu^2 \beta_x \beta_y)}\{\varphi^2[(2f_T + f_1)f_1 + f_2^2 + B_y h f_1] + \\ &\qquad + \mu\,\beta_x[(2f_T + f_1)f_1 + 4f_2^2 - B_x h f_1]\}. \end{aligned}\right\} \quad (4.152)$$

For brevity we have introduced the following notations

and also

$$\left.\begin{aligned} &\frac{1}{\beta_x} = 1 + \frac{Eh}{c_x a}, \qquad \frac{1}{\beta_y} = 1 + \frac{Eh}{c_y b}, \\ &B_x = \frac{32a}{\pi^4 h}\left[\frac{\varphi(\varphi\,\varkappa_x + \varkappa_y)(\varphi^2 - \mu)}{(1+\varphi^2)^2} - \varkappa_x\right], \\ &B_y = \frac{32b}{\pi^4 h}\left[\frac{(\varphi\,\varkappa_x + \varkappa_y)(1 - \mu\,\varphi^2)}{(1+\varphi^2)^2} - \varkappa_y\right], \\ &\varkappa_x = k_x a, \qquad \varkappa_y = k_y b. \end{aligned}\right\} \quad (4.153)$$

We now consider Equation (4.144) with the right-hand side (4.145). We shall try to find an approximate solution by Galerkin's method, substituting Expressions (4.149), (4.150), (4.151) and (4.152) as well as the expression for the known function of the temperature stresses $\Phi_T(x, y)$. Stipulating, as usual, that the result of the substitution must be orthogonal to each of the selected "coordinate" functions, we obtain a system of two second-order ordinary differential equations in which the time t plays the part of argument. Since the necessary calculations from now on rapidly increase in number, we introduce at this stage a few reasonable limitations. We shall confine our attention to a cylindrical shell, setting

$$x = x_0, \qquad \frac{\partial w_0}{\partial x} = 0, \qquad k_x = 0, \qquad k_y = \text{const.}$$

In addition, we shall assume that the temperature compressive force is uniformly distributed over the edges and that shear forces are absent, i.e.

$$N_x^T = -P_x = \text{const}, \qquad N_y^T = -P_y = \text{const.}$$

$$N_{xy}^T = 0.$$

Equation (4.144) then becomes

$$D\,\nabla^2\nabla^2 w_F + \varrho_0 h \frac{\partial^2 w_F}{\partial t^2} + \left(\varrho_0 h \varepsilon + \frac{\varkappa p_\infty}{c_\infty}\right)\frac{\partial w_F}{\partial t} + \\ + P_x \frac{\partial^2 w_F}{\partial x^2} + P_y \frac{\partial^2 w_F}{\partial y^2} + \mathscr{L}_1(w_F) = 0,$$

where $\mathfrak{L}_1(w_F)$ denotes the set of terms

$$\mathscr{L}_1(w) = -N_y^F k_y - \\ - \left[N_x^F \frac{\partial^2(w_T + w_F)}{\partial x^2} + N_y^F \frac{\partial^2(w_T + w_F)}{\partial y^2} + 2N_{xy}^F \frac{\partial^2(w_T + w_F)}{\partial x\,\partial y}\right] + \\ + \varkappa p_\infty M \frac{\partial w_F}{\partial x} + \frac{\varkappa(\varkappa + 1) p_\infty M^2}{4}\left[2\frac{\partial w_T}{\partial x}\,\frac{\partial w_F}{\partial y} + \left(\frac{\partial w_F}{\partial x}\right)^2\right] + \\ + \frac{\varkappa(\varkappa + 1) p_\infty M^3}{12}\left[3\left(\frac{\partial w_T}{\partial x}\right)^2 \frac{\partial w_F}{\partial x} + 3\frac{\partial w_T}{\partial x}\left(\frac{\partial w_F}{\partial x}\right)^2 + \left(\frac{\partial w_F}{\partial x}\right)^3\right].$$

The equations of Galerkin's method are obtained in the form

$$\frac{d^2 f_1}{dt^2} + \varepsilon \frac{df_1}{dt} + \Omega_1^2 f_1 + \frac{4}{a b \varrho_0 h}\int_0^a\!\!\int_0^b \mathscr{L}_1(w) \sin\frac{\pi x}{a} \sin\frac{\pi y}{b}\, dx\, dy = 0,$$

$$\frac{d^2 f_2}{dt^2} + \varepsilon \frac{df_2}{dt} + \Omega_2^2 f_2 + \frac{4}{a b \varrho_0 h}\int_0^a\!\!\int_0^b \mathscr{L}_1(w) \sin\frac{2\pi x}{a} \sin\frac{\pi y}{b}\, dx\, dy = 0.$$

Here Ω_1 and Ω_2 are the corresponding partial frequencies of natural small oscillations of a flat panel, namely

$$\Omega_1^2 = \frac{\pi^4 D}{a^4 \varrho_0 h}\left[(1 + \varphi^2)^2 - \frac{a^2}{\pi^2 D}(P_x + \varphi^2 P_y)\right],$$

$$\Omega_2^2 = \frac{\pi^4 D}{a^4 \varrho_0 h}\left[(4 + \varphi^2)^2 - \frac{a^2}{\pi^2 D}(4P_x + \varphi^2 P_y)\right].$$

We shall not, however, go further into the detailed calculations, which, although elementary, are extremely cumbersome. Instead, we shall simply give the final results. If we introduce the non-dimensional displacements and the non-dimensional time

$$\zeta_1 = \frac{f_1}{h}, \qquad \zeta_2 = \frac{f_2}{h}, \qquad \zeta_T = \frac{f_T}{h}, \qquad \tau = \Omega_0 t$$

together with the non-dimensional parameters

$$\left.\begin{aligned} &\frac{\Omega_j}{\Omega_0} = \omega_j; \quad \frac{1}{\Omega_0}\left(\varepsilon + \frac{\varkappa p_\infty}{\varrho h c_\infty}\right) = g, \quad \Omega_0 = \frac{\pi^2}{a^2}\sqrt{\frac{D}{\varrho_0 h}}, \\ &M\frac{h}{a} = \nu, \quad \frac{4\varkappa p_\infty}{\varrho_0 h^2 \Omega_0^2} = K, \quad \frac{\pi^4 E h^2}{16 \varrho_0 a^4 \Omega_0^2} = S, \end{aligned}\right\} \tag{4.154}$$

then the equations become[1]

$$\left.\begin{aligned} &\frac{d^2\zeta_1}{d\tau^2} + g\frac{d\zeta_1}{d\tau} + \omega_1^2\zeta_1 + \\ &+ K\left[-\frac{2}{3}\nu\zeta_2 + \frac{2}{9}\nu^2(\varkappa + 1)(\zeta_1 + \zeta_T)\zeta_1 + \right. \\ &\left. + \frac{56}{45}\nu^2(\varkappa + 1)\zeta_2^2 + \nu^3 b_{11}(\zeta_1 + \zeta_T)^2\zeta_2 + \nu^3 b_{12}\zeta_2^3\right] + \\ &+ S\zeta_1(c_{11}\zeta_1^2 + c_{12}\zeta_2^2) + \frac{16\varphi^2}{\pi^4\delta^2}S[(d_{11} + e_{11}\zeta_T + e_{14}\zeta_T^2)\zeta_1 + \\ &\qquad\qquad + (d_{12} + e_{12}\zeta_T)\zeta_1^2 + (d_{13} + e_{13}\zeta_T)\zeta_2^2] = 0, \\ &\frac{d^2\zeta_2}{d\tau^2} + g\frac{d\zeta_2}{d\tau} + \omega_2^2\zeta_2 + \\ &+ K\left[\frac{2}{3}\nu\zeta_1 + \frac{16}{45}\nu^2(\varkappa + 1)(\zeta_1 + \zeta_T)\zeta_2 + \nu b_{21}\zeta_1^3 + \right. \\ &\left. + \nu^3 b_{22}(\zeta_1 + \zeta_T)\zeta_2^2\right] + S\zeta_2(c_{12}\zeta_1^2 + c_{22}\zeta_2^2) + \\ &+ \frac{16\varphi^2}{\pi^4\delta^2}S[(d_{21} + e_{21}\zeta_T + e_{23}\zeta_T^2)\zeta_2 + (d_{22} + e_{22}\zeta_T)\zeta_1\zeta_2] = 0. \end{aligned}\right\} \tag{4.155}$$

---

[1] The explicit expressions for the coefficients of these equations can be found in the paper: BOLOTIN, V. V. Non-linear flutter of plates and shells. *Inzh. Sb.*, **28** (1960).

4.16. NON-LINEAR FLUTTER OF A FLAT PANEL. SOLUTION BY TRIGONOMETRIC SERIES

If the panel is flat ($\varkappa_y \equiv 0$) and there is no thermal buckling ($\zeta_T = 0$), then all the terms in Equations (4.155) which contain d_{ik} and e_{ik} disappear, and the terms containing b_{ik} can be simplified as follows:

$$\left.\begin{aligned}
&\frac{d^2\zeta_1}{d\tau^2} + g\frac{d\zeta_1}{d\tau} + \overset{\omega_1}{{}_\wedge}\zeta_1 + K\Big[-\frac{2}{3}\nu\zeta_2 + \frac{2}{9}\nu^2(\varkappa+1)\zeta_1^2 + \\
&\qquad + \frac{56}{45}\nu^2(\varkappa+1)\zeta_1^2 + \nu^3\zeta_2(b_{11}\zeta_1^2 + b_{12}\zeta_2^2)\Big] + \\
&\qquad\qquad + S\zeta_1(c_{11}\zeta_1^2 + c_{12}\zeta_2^2) = 0,\\
&\frac{d^2\zeta_2}{d\tau^2} + g\frac{d\zeta_2}{d\tau} + \omega_2^2\zeta_2 + K\Big[\frac{2}{3}\nu\zeta_1 + \frac{16}{45}\nu^2(\varkappa+1)\zeta_1\zeta_2 + \\
&\qquad + \nu^3\zeta_1(b_{21}\zeta_1^2 + b_{22}\zeta_2^2)\Big] + S\zeta_2(c_{21}\zeta_1^2 + c_{22}\zeta_2^2) = 0.
\end{aligned}\right\} \quad (4.156)$$

Here

$$\left.\begin{aligned}
&b_{11} = b_{21} = \frac{\pi^2}{40}(\varkappa+1), \qquad b_{12} = -\frac{9\pi^2}{70}(\varkappa+1),\\
&b_{22} = \frac{11\pi^2}{70}(\varkappa+1),\\
&c_{11} = 1 + \varphi^4 + \frac{2(\beta_x + 2\varphi^2\mu\beta_x\beta_y + \varphi^4\beta_y)}{1-\mu^2\beta_x\beta_y},\\
&c_{12} = c_{21} = 4(1+\varphi^4) + \frac{2(4\beta_x + 5\mu\varphi^2\beta_x\beta_y + \varphi^4\beta_y)}{1-\mu^2\beta_x\beta_y},\\
&c_{22} = 16 + \varphi^4 + \frac{2(16\beta_x + 8\mu\varphi^2\beta_x\beta_y + \varphi^4\beta_y)}{1-\mu^2\beta_x\beta_y}.
\end{aligned}\right\} \quad (4.157)$$

If both sides of the plate are exposed to the flow with equal velocities, then the system (4.156) assumes a simplified form, since all quadratic terms disappear:

$$\left.\begin{aligned}
&\frac{d^2\zeta_1}{d\tau^2} + g\frac{d\zeta_1}{d\tau} + \omega_1^2\zeta_1 + \\
&\qquad + \nu K\Big[-\frac{2}{3}\zeta_2 + \nu^3(b_{11}\zeta_1^2 + b_{12}\zeta_2^2)\Big] + \\
&\qquad\qquad + S\zeta_1(c_{11}\zeta_1^2 + c_{12}\zeta_2^2) = 0,\\
&\frac{d^2\zeta_2}{d\tau^2} + g\frac{d\zeta_2}{d\tau} + \omega_2^2\zeta_2 + \\
&\qquad + \nu K\Big[\frac{2}{3}\zeta_1 + \nu^3(b_{21}\zeta_1^2 + b_{22}\zeta_2^2)\Big] + \\
&\qquad\qquad + S\zeta_2(c_{21}\zeta_1^2 + c_{22}\zeta_2^2) = 0.
\end{aligned}\right\} \quad (4.158)$$

The parameter K now has twice its previous value.[1] The critical value of the parameter ν is given by the approximate formula

$$\nu_* = \frac{3}{4K}(\omega_2^2 - \omega_1^2), \tag{4.159}$$

which is derived from Formula (4.104) by taking into account that $\nu K = \beta$ in the notations of Section 4.10.

We shall try to find an approximate solution to the system of equations (4.158) in the form of periodic motion with finite amplitudes[2]

$$\zeta_1 = A_1 \cos\omega\tau + B_1 \sin\omega\tau + \cdots, \qquad \zeta_2 = A_2 \cos\omega\tau + \cdots \tag{4.160}$$

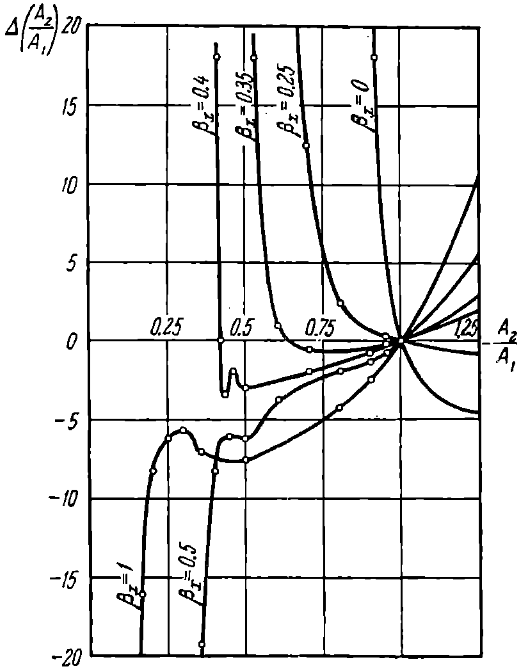


Fig. 134

---

[1] It is to be expected that when one side only is exposed to the flow, the effect of second-order terms will be small compared with the effect of third-order terms, and consequently Equations (4.158) can be considered as an approximate equation for this case also. This is confirmed by an analysis employing the small parameter method (Section 4.17).

[2] Bolotin, V. V., Gavrilov, Iu. V., Makarov, B. P. and Shveiko, Iu. Iu. Non-linear problems of the stability of flat panels at high supersonic velocities. *Izv. Akad. Nauk USSR, OTN, "Mekhanika i mashinostroenie"*, No. 3 (1959).

Here A_1, B_1, A_2, ω are unknown constants, the dots denote terms containing harmonics. We are considering a steady self-oscillatory domain and the initial phase is therefore non-essential. Substituting (4.160) into Equation (4.158) and ignoring terms containing harmonics, we obtain the following system of equations:

$$\left.\begin{aligned}
&(\omega_1^2-\omega^2)A_1+g\,\omega\,B_1-\frac{2}{3}\nu K A_2+\\
&\qquad+\frac{3}{4}\nu^3 K A_2\left[b_{11}\left(A_1^2+\frac{1}{3}B_1^2\right)+b_{12}A_2^2\right]+\\
&\qquad\qquad+\frac{3}{4}S A_1[c_{11}(A_1^2+B_1^2)+c_{12}A_2^2]=0,\\
&(\omega_1^2-\omega^2)B_1-g\,\omega\,A_1+\frac{1}{2}\nu^3 K b_{11} A_1 A_2 B_1+\\
&\qquad\qquad+\frac{3}{4}S B_1\left[c_{11}(A_1^2+B_1^2)+\frac{1}{3}c_{12}A_2^2\right]=0,\\
&(\omega_2^2-\omega^2)A_2+\frac{2}{3}\nu K A_1+\\
&\qquad+\frac{3}{4}\nu^3 K A_1[b_{21}(A_1^2+B_1^2)+b_{22}A_2^2]+\\
&\qquad\qquad+\frac{3}{4}S A_2\left[c_{21}\left(A_1^2+\frac{1}{3}B_1^2\right)+c_{22}A_2^2\right]=0,\\
&-g\,\omega\,A_2+\frac{2}{3}\nu K B_1+\frac{3}{4}\nu^3 K B_1\left[b_{21}(A_1^2+B_1^2)+\right.\\
&\qquad\qquad\left.+\frac{1}{3}b_{22}A_2^2\right]+\frac{1}{2}S\,c_{21}A_1A_2B_1=0.
\end{aligned}\right\}\quad(4.161)$$

We can find an approximate solution to this system if we assume that the damping is fairly slight, in which case $B_1^2 \ll A_1^2$, $A_2 \sim A_1$, $gB_1 \ll A_1$. Making use of the first two and the last equation, we can express A_1, B_1 and ω in terms of A_2 and then, if we substitute the values so found into the third equation, we obtain the resolvent $\Delta(A_2)=0$, the roots of which can be found graphically. Figure 134 shows the behavior of the function $\Delta(A_2/A_1)$ for a panel of infinite span with $\nu=\nu_*$. The calculations were carried out for $K=30.2$, $g=0.05$, $\varkappa=1.4$. For these values $\nu_*=0.372$.

The results for the same data are shown in Figs. 135 and 136. The most interesting result of these calculations is that for sufficiently small values of β_x (in the example considered this applies even for $\beta_x = 0.4$), periodic motion was possible for $v < v_*$. This can be seen from Fig. 134, where the branches of the curve $\Delta = \Delta(A_2/A_1)$ go off into the upper half-plane for values of $\beta_x \leqq 0.4$. The question of the stability of the solutions found has not been considered; however, from the form of the curves for $\beta_x \leqq 0.4$, by analogy with the results for a non-linear system with one degree of freedom, we conclude

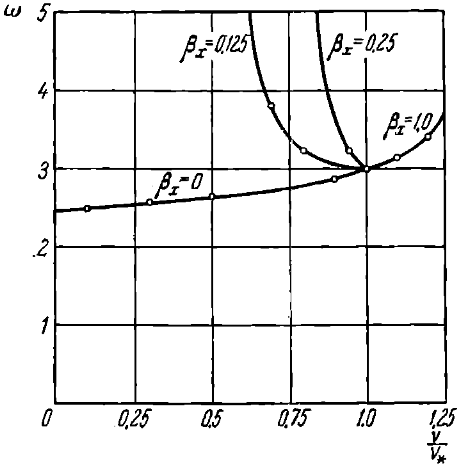


Fig. 135

that these solutions are unstable (on the same basis we can say that the branches going to the right from the point $v = v_*$ correspond to stable solutions). This conclusion has been fully confirmed by an analysis of the corresponding equations on electronic analogue computers (see below). Undoubtedly, in the region $v < v_*$ the physical system has stable limiting cycles resulting from non-linearities of any type which appear at large displacements (such as flow separation or a sharp increase in energy dissipation when the structure exceeds its elastic range, etc.). We cannot obtain such motions from Equations of the type (4.158) used here. Nevertheless, the unstable solutions obtained are of some value, since they indicate the order of the disturbances necessary to induce undamped oscillations of a panel at pre-critical velocities. The type of panel most sensitive to initial disturbances is one with free edges ($\beta_* = 0$). Since the maximum deflection of a plate is of the order of $A_1 h_1$ it follows from Fig. 136 that for $v = 0.9\, v_*$ a disturbance of the order of $0.5\, h$ is sufficient to induce dangerous oscillations. If $v = 0.8\, v_*$, the order of the necessary initial disturbance increases to h. If $v = 0.5\, v_*$, an initial disturbance of $2.5\, h$ is required. With increased stiffness of the longitudinal stiffeners the sensitivity of the panel to flutter at pre-critical velocities rapidly decreases, and starting at a certain value, "hard" excitation ceases to be possible. Thus the combined action of the panel as an element of the structure is an extremely useful factor. In the case of panels the edges of which are fixed against axial displacements "soft" excitation will exist with slowly increasing amplitudes in the post-critical region. The steady amplitudes are then of the order of the thickness of the panel (see the curve corresponding to $\beta_x = 1$ in Fig. 136).

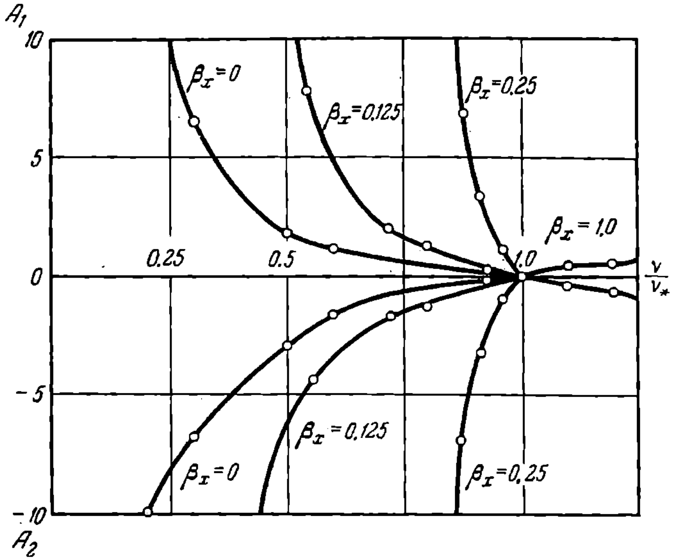


Fig. 136

If certain additional simplifications are introduced, we can derive a finite formula for the amplitudes A_1 and A_2. Of the four Equations (4.161) we rewrite the first and third omitting small terms containing gB_1 and B_1^2:

$$\left.\begin{aligned} &(\omega_1 - \omega^2) A_1 - \frac{2}{3}\nu K A_2 + \frac{3}{4}\nu^3 K A_2(b_{11} A_1^2 + b_{12} A_2^2) + \\ &\qquad + \frac{3}{4} S A_1(c_{11} A_1^2 + c_{12} A_2^2) = 0, \\ &(\omega_2^2 - \omega^2) A_2 + \frac{2}{3}\nu K A_1 + \frac{3}{4}\nu^3 K A_1(b_{21} A_1^2 + b_{22} A_2^2) + \\ &\qquad + \frac{3}{4} S A_2(c_{21} A_1^2 + c_{22} A_2^2) = 0. \end{aligned}\right\} \quad (4.162)$$

The two remaining equations take into account damping. If we linearize these equations we obtain

$$(\omega_1^2 - \omega^2) B_1 - g\,\omega A_1 = 0,$$
$$-g\,\omega A_2 + \frac{2}{3}\nu K B_1 = 0,$$

whence

$$A_2 = -\frac{2}{3}\,\frac{\nu K}{\omega^2 - \omega_1^2} A_1.$$

On the boundary of the stability region $\nu = \nu_*$ and $\omega = \omega_*$, where

$$\omega_*^2 = \frac{\omega_1^2 + \omega_2^2}{2}.$$

Consequently $A_2 \approx -A_1$. We can expect this relation to hold also for $\nu > \nu_*$, provided the difference $\nu - \nu_*$ is sufficiently small. This assumption is confirmed by more exact analyses (see, for example, Fig. 136).

In Equations (4.162), setting $A_2 = -A_1$, and eliminating the frequency ω, we arrive at the formula

$$A_1^2 \approx \frac{16}{9}\,\frac{\nu K - \frac{3}{4}(\omega_2^2 - \omega_1^2)}{S c_0 - \nu^3 K b_0},$$

where

$$\left.\begin{aligned} b_0 &= b_{21} + b_{22} - b_{11} - b_{12}, \\ c_0 &= c_{21} + c_{22} - c_{11} - c_{12}. \end{aligned}\right\} \quad (4.163)$$

If we denote

$$\sqrt[3]{\frac{S c_0}{K b_0}} = \nu_\infty, \qquad \frac{4}{3}\sqrt{\frac{K \nu_*}{S c_0}} = A_0, \quad (4.164)$$

then the formula assumes the very simple form

$$A = A_0 \sqrt{\frac{\dfrac{\nu}{\nu_*} - 1}{1 - \left(\dfrac{\nu}{\nu_\infty}\right)^3}}. \tag{4.165}$$

In the case of a "soft" disturbance $\nu_\infty > \nu_*$ on the stability boundary; in the case of a "hard" disturbance $\nu_\infty < \nu_*$. Figure 137 shows a comparison of the results given by Formula (4.165) and those obtained from the Equation $\Delta(A_2/A_1) = 0$ for the two extreme values of the coefficient β_x.

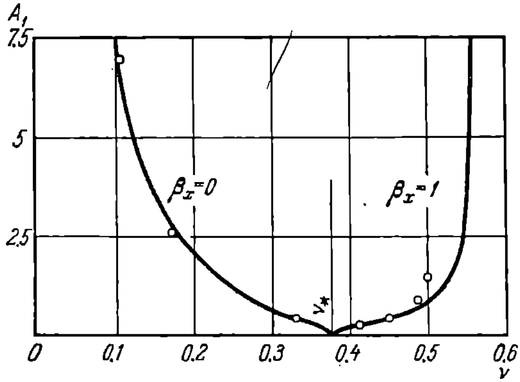


Fig. 137

In order to verify the approximate solutions obtained, and also to clarify the nature of their stability, the system of Equations (4.158) was solved on an electronic computer.[1] The solution was carried out for a panel of infinite span with edges on roller supports. The problem was to establish the nature of the disturbed solutions for various initial conditions. In all cases it was assumed that $\zeta_1'(0) = \zeta_2'(0) = 0$ and the values of $\zeta_1(0)$ and $\zeta_2(0)$ were varied so as to establish for given ν and β_* the boundary between the damped and infinitely increasing solutions.

Figure 138 shows the boundaries of the regions of stability for the example considered above ($\beta_x = 0$ and two-sided flow). The region of attraction of the trivial solution was located near the origin of coordinates; the boundaries of the region in the other three quadrants of the plane of $\zeta_1(0)$, $\zeta_2(0)$ were not determined. We see from the graph that as the critical value $\nu_* = 0.372$ is approached the size of the region of attraction of the trivial solution diminishes, and in the

---

[1] See reference on p. 291.

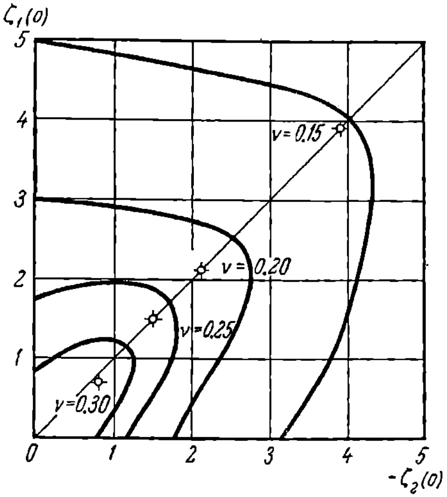


Fig. 138

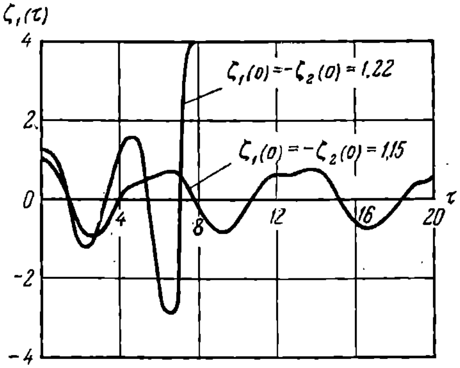


Fig. 139

limiting case the region disappears. On the same graph the points $\zeta_1(0) = A_1$, $\zeta_2(0) = A_2$ are plotted, corresponding to values of the approximate solution (4.160) with $\tau = 0$.

Typical solutions in the neighborhood of the boundary of the stability region are shown in Figs. 139 and 140. Figure 139 corresponds to the case of $\nu = 0.3$. When $\zeta_1(0) = -\zeta_2(0) = 1.15$ the initial disturbance, although small, is damped out with time. The small increase in the initial disturbance to $\zeta_1(0) = -\zeta_2(0) = 1.22$ causes a distinct change in the nature of the oscillations: flutter occurs with rapidly increasing amplitudes. A similar behavior of the system is

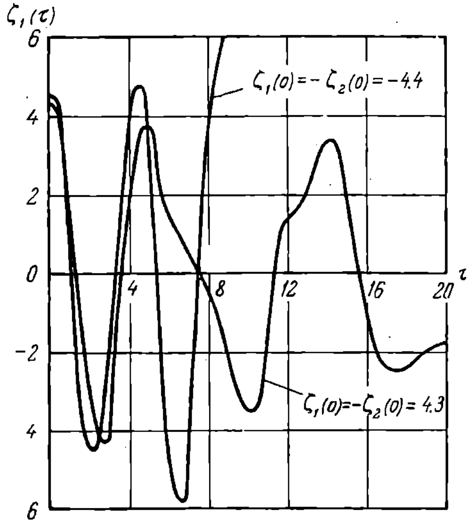


Fig. 140

observed for $\nu = 0.15$ (Fig. 140). Here transition from stability to instability occurs with increase in the initial disturbance from $\zeta_1(0) = -\zeta_2(0) = 4.3$ to $\zeta_1(0) = -\zeta_2(0) = 4.4$.

To conclude we emphasize once more that these results refer to cylindrical bending of a plate with edges on roller supports, i.e. to the case when geometric non-linearity caused by the action of forces in the middle surface is absent. Actual panels often have considerable geometric non-linearity and for these "soft" excitation of flutter is to be expected.

4.17. SMALL PARAMETER METHOD FOR INVESTIGATION OF NON-LINEAR FLUTTER

So far, in order to find steady periodic solutions to systems of the type (4.141), we have used simplified approximate methods. We are mainly concerned with the behavior of the system near the boundary of the flutter region, for which case a theory can be evolved on the basis of the small-parameter method. We write the system of Equations

(4.141) in the form

$$\frac{d^2\zeta_j}{d\tau^2} + g\frac{d\zeta_j}{d\tau} + \omega_j^2\zeta_j + v K \sum_{k=1}^{n} b_{jk}(\nu)\zeta_k = \Psi_j(\zeta_1, \zeta_2, \ldots, \zeta_n, \nu)$$

$$(j = 1, 2, \ldots, n). \qquad (4.166)$$

Here b_{jk} are analytic functions of the parameter ν, ψ_j are analytic (non-linear) functions of the generalized coordinates ζ_k and the parameter ν. The corresponding linear system

$$\frac{d^2\zeta_j}{d\tau^2} + g\frac{d\zeta_j}{d\tau} + \omega_j^2\zeta_j + \nu K \sum_{k=1}^{n} b_{jk}(\nu)\zeta_k = 0$$

$$(j = 1, 2, \ldots, n) \qquad (4.167)$$

has a solution in the form $\zeta_j = Z_j e^{\sigma\tau}$. For sufficiently small values of ν all the characteristic exponents σ lie in the left-hand half-plane of the complex variable and the trivial solution of System (4.116), $\zeta_1 = \zeta_2 = \cdots = \zeta_n = 0$ is asymptotically stable with respect to small disturbances. The value of the parameter $\nu = \nu_*$ at which two of the characteristic exponents become purely imaginary and the remainder, as before, lie in the left-hand half-plane, are critical values and correspond to the critical velocity of panel flutter in the usual (linear) presentation of this problem.

We shall try to find periodic solutions to System (4.166) in the neighborhood of the critical value. Let $\nu = \nu_* + \eta\,\nu_1$ where ν_1 is of the same order as ν, and η is a small parameter. Taking Equations (4.167) as a degenerating system when $\nu = \nu_*$ and including the small multiplier η in the non-linear functions, we obtain

$$\frac{d^2\zeta_j}{d\tau^2} + g\frac{d\zeta_j}{d\tau} + \omega_j^2\zeta_j + \nu_* K \sum_{k=1}^{n} b_{jk}(\nu_*)\zeta_k =$$

$$= -\eta\,\nu_1 K \sum_{k=1}^{n} b_{jk}(\nu_* + \eta\,\nu_1)\zeta_k - \nu_* K \sum_{k=1}^{n} [b_{jk}(\nu_* + \eta\,\nu_1) -$$

$$- b_{jk}(\nu_*)]\zeta_k + \eta\,\Psi_j(\zeta_1, \zeta_2, \ldots, \zeta_n, \nu_* + \eta\,\nu_1)$$

$$(j = 1, 2, \ldots, n). \qquad (4.168)$$

Bearing in mind that when $\nu = \nu_*$ the degenerating system (4.167) has two periodic linearly independent solutions

$$\zeta_j = \varphi_{j1}(\tau), \qquad \zeta_j = \varphi_{j2}(\tau) \qquad (4.169)$$

with simple characteristic exponents $\sigma = \pm i\,\omega_*$ (ω_* is the non-dimensional flutter frequency, (see Fig. 141 a)), we introduce the time transformation

$$\omega_*\tau(1 + \eta\,p_1 + \eta^2 p_2 + \cdots) = \tau_1$$

and seek a solution to the non-linear system (4.168) in the form

$$\zeta_1(\tau_1) = A\,\varphi_{11}(\tau_1) + \eta\,\zeta_1^{(1)}(\tau_1) + \eta^2\,\zeta_1^{(2)}(\tau_1) + \cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\zeta_n(\tau_1) = A\,\varphi_{n1}(\tau_1) + \eta\,\zeta_n^{(1)}(\tau_1) + \eta^2\,\zeta_n^{(2)}(\tau_1) + \cdots$$

Here A is the first approximation amplitude. $p_1, p_2, \cdots$ are corrections to the frequency of oscillations[1].

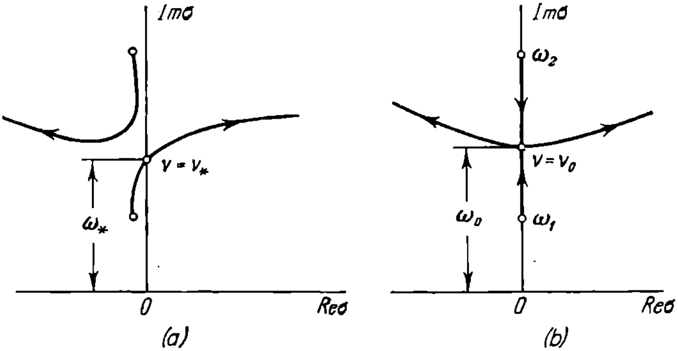


Fig. 141

We substitute these expressions into Equations (4.168) and equate coefficients of like powers of the small parameter η:

$$\left.\begin{aligned}
&\omega_*^2\frac{d^2\varphi_{j1}}{d\tau_1^2} + g\,\omega_*\frac{d\varphi_{j1}}{d\tau_1} + \omega_j^2\varphi_{j1} + \nu_* K\sum_{k=1}^{n} b_{jk}(\nu_*)\,\varphi_{k1} = 0,\\
&\omega_*^2\frac{d^2\zeta_j^{(1)}}{d\tau_1^2} + g\,\omega_*\frac{d\zeta_j^{(1)}}{d\tau_1} + \omega_j^2\zeta_j^{(1)} + \nu_* K b_{jk}(\nu_*)\,\zeta_k^{(1)} =\\
&= -2p_1\,\omega_*^2 A\frac{d^2\varphi_{j1}}{d\tau_1^2} - p_1 g\,\omega_* A\frac{d\varphi_{j1}}{d\tau_1} - \nu_1 K\sum_{k=1}^{n} b_{jk}(\nu_*)\,\varphi_{k1} -\\
&\quad - \nu_*\nu_1 K\sum_{k=1}^{n}\frac{\partial b_{jk}(\nu_*)}{\partial\nu_*} + \Psi_j(A\varphi_{11}, A\,\varphi_{21}, \ldots, A\,\varphi_{n1}, \nu_*)\\
&\qquad (j = 1, 2, \ldots, n).
\end{aligned}\right\}\tag{4.170}$$

---

[1] In the present case the introduction of a small parameter signifies that we are examining the behavior of the system sufficiently close to the boundary of the flutter region and that the amplitudes of the non-linear oscillations are assumed to be fairly small. The parameter η indicates the smallness of the various terms; after the necessary calculations have been carried out we can return to the initial equations, setting $\eta = 1$, $\nu_1 = \nu - \nu_*$.

The first equation is satisfied by virtue of the chosen degenerating solution (4.169); the second and subsequent equations are of the form

$$\omega_*^2 \frac{d^2\zeta_j^{(s)}}{d\tau_1^2} + g\,\omega_* \frac{d\zeta_j^{(s)}}{d\tau_1} + \omega_j^2 \zeta_j^{(s)} + \nu_* K \sum_{k=1}^{n} b_{jk}(\nu_*)\,\zeta_k^{(s)} =$$
$$= \Psi_j^{(s)}(A, \zeta_1^{(1)}, \zeta_2^{(1)}, \dots, \zeta_n^{(s-1)}, p_1, p_2, \dots, p_s, \nu_1, \tau_1)$$
$$(j = 1, 2, \dots, n; \quad s = 1, 2, \dots). \tag{4.171}$$

The constants A, p_1, p_2, $\cdots$, p_s must be chosen in such a way that Equations (4.171) have periodic solutions with period 2π.

Let $\psi_{j1}(\tau_1)$ and $\psi_{j2}(\tau_2)$ be periodic linearly independent solutions to the system, adjoint with respect to the system (4.167), and corresponding to the characteristic exponents $\pm i\,\omega_*$. The conditions of periodicity of the system (4.171) for $s = 1$ are

$$\sum_{j=1}^{n} \int_0^{2\pi} \Psi_j^{(1)}(A, p_1, \nu_1, \tau_1)\, \psi_{j1}(\tau_1)\, d\tau_1 = 0,$$

$$\sum_{j=1}^{n} \int_0^{2\pi} \Psi_j^{(1)}(A, p_1, \nu_1, \tau_1)\, \psi_{j2}(\tau_1)\, d\tau_1 = 0.$$

From these we find the constants A and p_1, and the solution to the system will be $\zeta_k^{(1)} = A_1\, \varphi_{k1}(\tau_1) + \xi_k^{(1)}(\tau_1)$, where A_1 is a new constant (a correction to the amplitude of the degenerating solution), $\xi_k^{(1)}(\tau_1)$ is some particular integral of System (4.171) for $s = 1$ and for the values of A and p_1 previously found. Substituting these expressions into the equations of the next approximation and applying the periodicity conditions, we obtain the following equations for finding A_1 and p_2:

$$\sum_{j=1}^{n} \int_0^{2\pi} \Psi_j^{(2)}(A_1, p_2, \nu_1, \tau_1)\, \psi_{j1}(\tau_1)\, d\tau_1 = 0,$$

$$\sum_{j=1}^{n} \int_0^{2\pi} \Psi_j^{(2)}(A_1, p_2, \nu_1, \tau_1)\, \psi_{j2}(\tau_1)\, d\tau_1 = 0.$$

This process can be continued, and a sufficiently small value of the parameter η can always be chosen such that the process converges.

In establishing actual periodicity conditions we use the following method. We consider the determinant of the system of linear differential equations which result from substituting the expressions $\varphi_{j1} = Z_{j1} \exp(i\,\tau_1)$, into the degenerate Equations (4.170):

$$\Delta = \left| (\omega_j^2 - \omega_*^2 + i\,g\,\omega_*)\,\delta_{jk} + \nu_* K\, b_{jk}(\nu_*) \right|. \tag{4.172}$$

Here δ_{jk} is the Kronecker delta. The rank of the determinant (4.172) is obviously $n - 1$, since the system (4.170) has a single periodic solution with a simple characteristic exponent equal to i. We express the right-hand sides of Equations (4.171) in the form

$$\begin{aligned} &\Psi_j^{(s)}(A, \zeta_1^{(1)}, \zeta_2^{(1)}, \dots, \zeta_n^{(s-1)}, p_1, p_2, \dots, p_s, \nu_1, \tau_1) = \\ &= R_{j1}^{(s)} e^{i\tau_1} + R_{j2}^{(s)} e^{-i\tau_1} + \cdots \\ &\qquad (j = 1, 2, \dots, n;\ s = 1, 2, 3, \dots) \end{aligned} \tag{4.173}$$

(the dots denote terms containing higher harmonics). Replacing one of the columns of the determinant (4.172) by a column of the Fourier coefficients $R_{j1}^{(s)}$ $(j = 1, 2, \dots, n)$, equating the resulting determinant to zero:

$$\Delta_0^{(s)} = \Delta_1^{(s)} + i\Delta_2^{(s)} = 0, \tag{4.174}$$

and separating the real and imaginary parts, we obtain the periodicity conditions,

$$\left.\begin{aligned} \operatorname{Re}\Delta_0^{(s)} &= \Delta_1^{(s)}(A_{s-1}, p_s, \nu_1) = 0, \\ \operatorname{Im}\Delta_0^{(s)} &= \Delta_2^{(s)}(A_{s-1}, p_s, \nu_1) = 0 \\ &(s = 1, 2, \dots). \end{aligned}\right\} \tag{4.175}$$

We note that if the other periodic solution $\varphi_{j2} = Z_{j2}\exp(-i\tau_1)$ is substituted into Equations (4.171), then by virtue of the fact that the coefficients $R_{j1}^{(s)}$ and $R_{j2}^{(s)}$ in Formulae (4.173) are complex conjugate, we arrive at a determinant complex conjugate with respect to (4.174), i.e. at the same periodicity Conditions (4.175).

Certain remarks should be made concerning the choice of the degenerate system. Since in actual problems the damping g is usually a small quantity, it might be expedient to take as the degenerate system one which describes the behavior of a corresponding non-dissipative system:

$$\left.\begin{aligned} \frac{d^2\zeta_j}{d\tau^2} + \omega_j^2\zeta_j + \nu K \sum_{k=1}^{n} b_{jk}(\nu)\zeta_k = 0 \\ (j = 1, 2, \dots, n). \end{aligned}\right\} \tag{4.176}$$

If the value of ν is small enough, all the characteristic exponents of this system remain purely imaginary. At a certain value ν_0 there appear two pairs of multiple characteristic exponents equal to $i\,\omega_0$ and $-i\,\omega_0$ (see Fig. 141 b, where only the upper half-plane is shown). With further increase in the parameter ν, characteristic exponents with positive real parts appear. In the majority of studies in the field of elastic panel flutter at high supersonic velocities the effect of dissipative terms is not taken into account; beyond the critical value of the parameter (and correspondingly, beyond the critical flutter velocity) a value is taken at which multiple characteristic exponents first appear. If, however, taking (4.176) as the degenerating system with $\nu = \nu_0$, we attempt to find a periodic solution to the corresponding non-linear system in the neighborhood of $\nu = \nu_0$ in the form of a series in powers of the small parameter, we encounter serious difficulties characteristic, apparently, of other flutter problems. When $\nu = \nu_0$, the degenerate system (4.176) has two-fold characteristic exponents $i\,\omega_0$ and $-i\,\omega_0$, but the corresponding elementary divisors are non-linear. Therefore the Poincaré functional determinant with such a choice of a degenerate system vanishes.

The difficulties can be overcome if, as was done above, (4.176) is taken as the degenerate system. In this case, when $\nu = \nu_*$, we have simple characteristic exponents (Fig. 141 a). If there is only slight damping it is expedient to retain in the solution only those terms which contain g to a power not higher, for example, than the first. In some cases it is possible to find solutions with a "vanishing small damping": damping is taken into account in the construction of the degenerate solution and the periodicity conditions, after which, since it is small, it is not included in the final solution. The particular effect of damping and the difference in behavior of the two systems, one of which is considered without account being taken of damping, and the other with a vanishingly small damping, have already been indicated (in Section 1.16).

We shall illustrate the order of the calculations by taking as an example the system of equations

$$\left.\begin{aligned} &\frac{d^2\zeta_1}{d\tau^2} + g\frac{d\zeta_1}{d\tau} + \omega_1^2\zeta_1 - \frac{2}{3}\nu K\zeta_2 + \Psi_1(\zeta_1, \zeta_2) = 0, \\ &\frac{d^2\zeta_2}{d\tau_2} + g\frac{d\zeta_2}{d\tau} + \omega_2^2\zeta_2 + \frac{2}{3}\nu K\zeta_1 + \Psi_2(\zeta_1, \zeta_2) = 0, \end{aligned}\right| \qquad (4.177)$$

which describes the oscillations of a flat unbuckled panel. By considering the characteristic equation of the corresponding linear system

$$\sigma^4 + 2g\,\sigma^3 + (\omega_1^2 + \omega_2^2 + g^2)\,\sigma^2 + g(\omega_1^2 + \omega_2^2)\,\sigma + \\ + \omega_1^2\,\omega_2^2 + \frac{4}{9}\nu^2 K^2 = 0,$$

it can easily be shown that this system has periodic solutions with a frequency $\omega_* = \omega_0$ when $\nu_* = \nu_0 + O(g^2)$, where

$$\nu_0 = \frac{3}{4}\,\frac{\omega_2^2 - \omega_1^2}{K}, \qquad \omega_0 = \sqrt{\frac{\omega_1^2 + \omega_2^2}{2}}.$$

These solutions are

$$\left.\begin{aligned} \varphi_{11} &= \cos\tau_1 + O(g^2), \\ \varphi_{21} &= -\cos\tau_1 - g\,\chi\,\sin\tau_1 + O(g^2), \\ \varphi_{12} &= \sin\tau_1 + O(g^2), \\ \varphi_{22} &= -\sin\tau_1 + g\,\chi\cos\tau_1 + O(g^2), \end{aligned}\right\} \tag{4.178}$$

where

$$\chi = \frac{\sqrt{2} \times \sqrt{(\omega_2^2 + \omega_1^2)}}{\omega_2^2 - \omega_1^2}.$$

From now on we shall not in general write down terms containing g to a power higher than the first[1]. In particular we shall accept that $\nu_* = \nu_0$.

We write down the determinant (4.172) for the system (4.177):

$$\Delta = \begin{vmatrix} \omega_1^2 - \omega_0^2 + i\,g\,\omega_0 & -\dfrac{2}{3}\nu_* K \\ \dfrac{2}{3}\nu_* K & \omega_2^2 - \omega_0^2 + i\,g\,\omega_0 \end{vmatrix}.$$

The periodicity Condition (4.174) now becomes

$$(\omega_1^2 - \omega_0^2 + i\,g\,\omega_0)\,R_{21}^{(s)} - \frac{2}{3}\nu_* K R_{11}^{(s)} = 0, \tag{4.179}$$

where $R_{21}^{(s)}$ and $R_{11}^{(s)}$ are Fourier coefficients of $\exp(i\tau_1)$ for the functions $\Psi_1^{(s)}$ and $\Psi_2^{(s)}$. If the functions $\Psi_1^{(s)}$ and $\Psi_2^{(s)}$ are written in real form, then

$$\Psi_1^{(s)} = P_1^{(s)}\cos\tau_1 + Q_1^{(s)}\sin\tau_1 + \cdots,$$
$$\Psi_2^{(s)} = P_2^{(s)}\cos\tau_1 + Q_2^{(s)}\sin\tau_1 + \cdots,$$

whence

$$R_{11}^{(s)} = \frac{P_1^{(s)} - i\,Q_1^{(s)}}{2}, \qquad R_{21}^{(s)} = \frac{R_2^{(s)} - i\,Q_2^{(s)}}{2}. \tag{4.180}$$

Substituting (4.180) into Equation (4.179) and separating the real and imaginary parts, we obtain

$$P_1^{(s)} + P_2^{(s)} - g\,\chi\,Q_2^{(s)} = 0, \qquad Q_1^{(s)} + Q_2^{(s)} + g\,\chi\,P_2^{(s)} = 0. \tag{4.181}$$

These are the required periodicity conditions.

---

[1] It is assumed here that the partial natural frequencies are sufficiently far apart. In the case of multiple or very close natural frequencies the effect of damping becomes very important, as can be seen from the formula for ν_*.

Proceeding now to the calculations, we take the non-linear functions Ψ_1 and Ψ_2 for the case of a flat unbuckled panel in the form

$$\left.\begin{aligned}\Psi_1(\zeta_1,\zeta_2,\nu) &= \nu^2 K\left[\frac{2}{9}(\varkappa+1)\zeta_1^2+\frac{56}{45}(\varkappa+1)\zeta_2^2+\right.\\ &\left.\quad+\nu\zeta_2(b_{11}\zeta_1^2+b_{12}\zeta_2^2)\right]+S\zeta_1(c_{11}\zeta_1^2+c_{12}\zeta_2^2),\\ \Psi_2(\zeta_1,\zeta_2,\nu) &= \nu^2 K\left[\frac{16}{45}(\varkappa+1)\zeta_1\zeta_2+\nu\zeta_1(b_{21}\zeta_1^2+b_{22}\zeta_2^2)\right]+\\ &\quad+S\zeta_2(c_{21}\zeta_1^2+c_{22}\zeta_2^2).\end{aligned}\right\} \tag{4.182}$$

Bearing in mind (4.178), we try to find a periodic solution in the form

$$\left.\begin{aligned}\zeta_1 &= A\cos\tau_1+\eta\zeta_1^{(1)}+\eta^2\zeta_2^{(2)}+\cdots,\\ \zeta_2 &= -A(\cos\tau_1+g\chi\sin\tau_1)+\eta\zeta_2^{(1)}+\eta^2\zeta_2^{(2)}+\cdots\end{aligned}\right\} \tag{4.183}$$

After substituting Expressions (4.183) into Formulae (4.182), we obtain

$$P_1 = 2p_1\omega_*^2 A-\frac{2}{3}\nu_1 KA+\frac{3}{4}\nu_*^3 KA^3(b_{11}+b_{12})-\frac{3}{4}SA^3(c_{11}+c_{12}),$$

$$Q_1 = p_1 g\omega_*^2 A-\frac{2}{3}\nu_1 KAg\chi+\frac{1}{4}\nu_*^3 KA^3(b_{11}+3b_{12})-\frac{1}{2}SA^3 g\chi c_{12},$$

$$\begin{aligned}P_2 = &-2p_1\omega_*^2 A-\frac{2}{3}\nu_1 KA-\frac{3}{4}\nu_*^3 KA^3(b_{21}+b_{22})+\\ &+\frac{3}{4}SA^3(c_{21}+c_{22}),\end{aligned}$$

$$\begin{aligned}Q_2 = &-2p_1 g\chi\omega_*^2 A-p_1 g\omega_* A-\frac{1}{2}\nu_*^3 KA^3 g b_{22}\chi+\\ &+\frac{1}{4}SA^3 g\chi(c_{21}+3c_{22}).\end{aligned}$$

The first of the periodicity Conditions (4.181) for the case when $g^2 \ll 1$ now becomes

$$\begin{aligned}-\frac{4}{3}\nu_1 KA+\frac{3}{4}\nu_*^3 KA^3(b_{11}+b_{12}-b_{21}-b_{22})+&\\ +\frac{3}{4}SA^3(c_{21}+c_{22}-c_{11}-c_{12})&=0.\end{aligned}$$

Thus the amplitude to a first approximation is

$$A = \frac{4}{3}\sqrt{\frac{(\nu - \nu_*) K}{S c_0 - \nu_*^3 K b_0}}, \tag{4.184}$$

where the definitions of (4.163) have been used. The first correction to the frequency p_1 is determined from the second periodicity condition of (4.181).

It should be noted that the coefficients of quadratic terms do not appear in Formula (4.184). This means that to a first approximation they have no effect on the amplitude of steady flutter in the neighborhood of the stability boundary.

4.18. Analysis of Results

Let us examine Formula (4.184) from the point of view of the behavior of the system close to the flutter boundary. If $\partial A/\partial \nu > 0$ for $\nu \geqq \nu_*$, then this means that "soft" excitation exists. If for $\nu > \nu_*$ there are no real values of A, and for $\nu \leqq \nu_*$ the inequality $\partial A/\partial \nu < 0$ holds, then this corresponds to "hard" excitation of flutter. A solution for which $\partial A/\partial \nu < 0$ is evidently unstable: it can be looked upon as a sort of "water divide" which separates the region of stability attraction of the trivial solution ($A = 0$).

Using the definitions of (4.164), we can write Formula (4.184) in a form analogous to (4.165):

$$A = A_0 \sqrt{\frac{\dfrac{\nu}{\nu_*} - 1}{1 - \left(\dfrac{\nu_*}{\nu_\infty}\right)^3}}. \tag{4.185}$$

The formulae give results which are more nearly the same, the closer ν is to ν_*. Although Formula (4.165) was given without a strict justification, its limits of application can evidently be extended. An alternative form of Formula (4.185) is

$$A = A_0 \sqrt{\frac{\dfrac{M}{M_*} - 1}{1 - \left(\dfrac{M_*}{M_\infty}\right)^3}}, \tag{4.186}$$

where

$$\left.\begin{aligned} M_* &= \frac{3}{4}\frac{a}{h}\frac{\omega_2^2-\omega_1^2}{K}, \qquad M_\infty = \frac{a}{h}\sqrt[3]{\frac{S c_0}{K b_0}}, \\ A_0 &= \frac{4}{3}\sqrt{\left(\frac{K M_*}{S c_0}\frac{h}{a}\right)}. \end{aligned}\right\} \tag{4.187}$$

Excitation of flutter will obviously be "soft" if $M_\infty > M_*$ and "hard" if $M_\infty < M_*$.

We shall carry out the calculations for a square flat panel with $\mu = 0.25$, $\varkappa = 1.4$, $P_x = P_y = 0$ and with two extreme assumptions concerning the rigidity of the stiffening structure. If the structure offers no resistance to relative displacement of the edges, we have the case of $c_x = c_y = 0$, and, as follows from Formulae (4.153), $\beta_x = \beta_y = 0$. The opposite assumption corresponds to the case of an absolutely rigid stiffening structure. If $c_x \to \infty$, $c_y \to \infty$, it can be seen from Formulae (4.153) that $\beta_x = \beta_y = 1$. We shall confine our attention simply to these two cases, defining the first as the case of an absolutely flexible frame and the second as the case of an absolutely rigid frame.[1]

In the case of an absolutely flexible frame calculations based on Formulae (4.157) give $c_{11} = 2.00$, $c_{12} = c_{21} = 11.24$, $c_{22} = 17.00$, and consequently $c_0 = 15.00$. In the case of an absolutely rigid frame, Formulae (4.157) give $c_{11} = 7.34$, $c_{12} = c_{21} = 24.44$, $c_{22} = 57.50$, whence $c_0 = 50.16$. Also, taking into account that $b_{11} = b_{21} = 0.59$, $b_{12} = -3.04$, $b_{22} = 3.72$, we find $b = 6.76$.

From Formulae (4.154) we find

$$K = \frac{48(1-\mu^2)}{\pi^4}\frac{\varkappa p_\infty}{E}\left(\frac{a}{h}\right)^4, \qquad S = \frac{3}{4}(1-\mu^2).$$

Thus for a steel panel ($E = 2 \times 10^6$ kg/cm^2, $\mu = 0.25$) with $p_\infty = 1$ kg/cm, $\varkappa = 1.4$, we obtain

$$K = 32.7\left(\frac{a}{h}\right)^4, \qquad S = 0.682$$

---

[1] Elastic reactions can be induced, of course, not only by the frame, but also from other elements which resist deformation of the panel. Moreover, we are considering the stiffness of the supporting structure only in a tangential direction; in a normal direction its stiffness is always assumed to be large enough for the panel to be considered as resting on a rigid contour.

and consequently, from Formulae (4.187),

$$\left.\begin{aligned} M_* &= 0.48 \times 10^8 \left(\frac{h}{a}\right)^3, \\ M^{\mathrm{I}}_\infty &= 168 \left(\frac{h}{a}\right)^{1/3}, \\ M^{\mathrm{II}}_\infty &= 251 \left(\frac{h}{a}\right)^{1/3}, \\ A^{\mathrm{I}}_0 &= 1.63, \quad A^{\mathrm{II}}_0 = 0.89 \end{aligned}\right\} \tag{4.188}$$

(the superscript I refers to the case of edges free to move, and II indicates fixed edges).

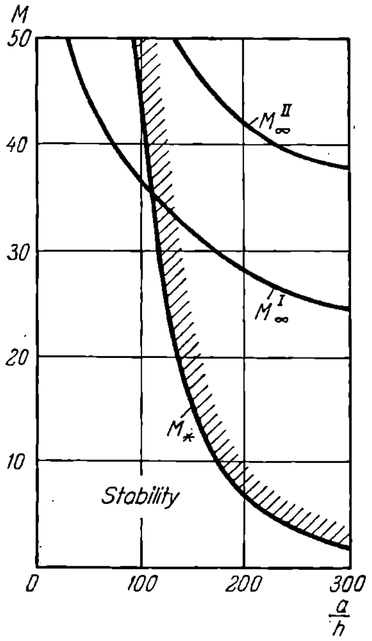


Fig. 142

The Mach numbers M_*, M^{I}_∞ and M^{II}_∞ are plotted as functions of h/a in Fig. 142. "Hard" excitation of flutter, even in the case of edges free to move, is possible only at extremely high Mach numbers. We recall that here, in contrast to Section (4.16), we are considering a panel which is square in plan and which is undergoing bending in two directions, i.e. a system which possesses sufficiently large geometric non-linearity.

Figure 143 shows the variation of amplitude with increase in Mach number beyond the critical value M_* for various ratios a/h. It was plotted from Formula (4.165), which has a wider range of application than Formula (4.185), which corresponds to a first approximation by the small parameter method. If we pass from an absolutely rigid to an absolutely flexible frame, the amplitude of oscillations is approximately doubled. Actual boundary conditions would correspond to some intermediate case.

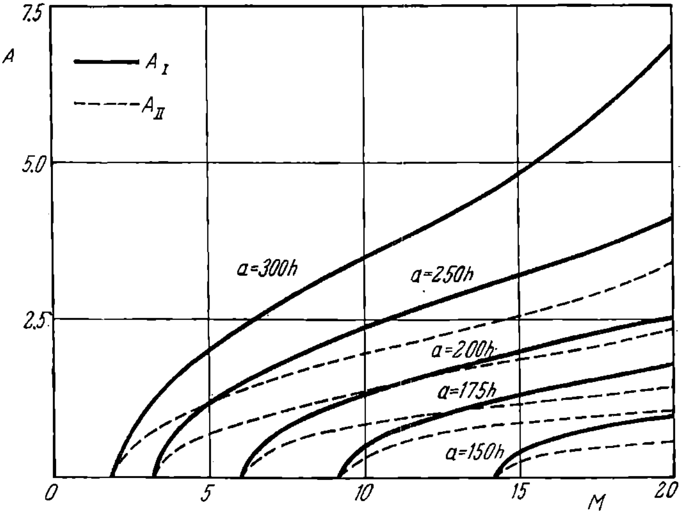


Fig. 143

Finally, we estimate the order of the stresses induced in the panel after the critical flutter velocity is exceeded. We evaluate the quantity

$$\sigma_x = -\frac{Eh}{2(1-\mu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \mu \frac{\partial^2 w}{\partial y^2}\right) + \frac{N_x}{h}$$

along the line $y = b/2$. Noting that

$$w\left(x, \frac{b}{2}, t\right) = f_1 \sin\frac{\pi x}{a} + f_2 \sin\frac{2\pi x}{a},$$

$$N_x\left(x, \frac{b}{2}, t\right) =$$

$$= \frac{\pi^2 E h \varphi^4}{a^2}\left[\frac{9 f_1 f_2}{(1+4\varphi^2)^2}\cos\frac{\pi x}{a} - \frac{f_1 f_2}{(9+4\varphi^2)^2}\cos\frac{3\pi x}{a} + \frac{f_1^2 + 4 f_2^2}{8\varphi^4}\right] +$$

$$+ \frac{\pi^2 E h \beta_x}{8a^2(1-\mu^2\beta_x\beta_y)}\left[f_1^2 + 4f_2^2 + \mu\beta_y(f_1^2 + f_2^2)\right]$$

and taking into account Formulae (4.183), we obtain

$$\max_t \sigma_x \approx \frac{\pi^2 E h^2}{a^2}\left[\frac{A\, G_1(x)}{2(1-\mu^2)} + A^2\, G_2(x)\right], \qquad (4.189)$$

where

$$G_1(x) = -(4+\mu\,\varphi^2)\sin\frac{2\pi x}{a} + (1+\mu\,\varphi^2)\sin\frac{\pi x}{a},$$

$$G_2(x) = \frac{5}{8} + \frac{\beta_x(5+2\mu\,\beta_y)}{8(1-\mu^2\beta_x\beta_y)} - $$

$$- \frac{9\varphi^2}{(1+4\varphi^2)^2}\cos\frac{\pi x}{a} + \frac{\varphi^4}{(9+4\varphi^2)^2}\cos\frac{3\pi x}{a}.$$

The functions G_1 and G_2 for the case of $\varphi = 1, \mu = 0.25$ are plotted in Fig. 144. For values of A which are not too large, for example

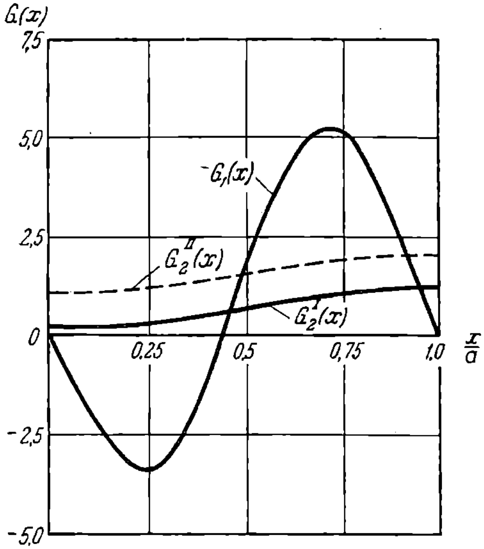


Fig. 144

for $A \leqq 3$, the maximum value of the function in the square brackets of Formula (4.189) differs only slightly from its value at $x = {}^3/_4 a$. We thus obtain the following very simple formulae:

$$\max \sigma_x^{\mathrm{I}} \approx \frac{\pi^2 E h^2}{a^2}(2.74\,A + 0.99\,A^2),$$

$$\max \sigma_x^{\mathrm{II}} \approx \frac{\pi^2 E h^2}{a^2}(2.74\,A + 1.62\,A^2).$$

The relation between the stress and Mach number M for various values of h/a is plotted in Fig. 145. The solid lines, as in the previous diagrams, represent the stress variation for the case of an absolutely flexible frame, and the broken lines correspond to an absolutely rigid frame. It will be seen from Fig. 145 that even if the critical Mach number, as determined by the methods of the linear theory, is exceeded by a considerable amount, the induced stresses are not extremely high. For a panel with a ratio of length of side to thickness equal to 250, the critical Mach number is

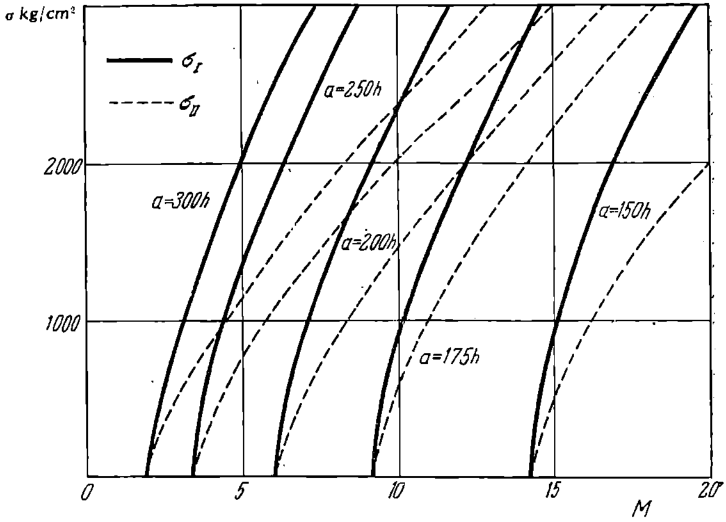


Fig. 145

$$M_* \approx 3.14.$$

If $M = 5$ the maximum stresses do not exceed 2000 kg/cm² even in the most unfavorable case of an absolutely flexible frame.

In this connection the question arises as to the possibility of the system being momentarily in the region of panel flutter without dangerous cracks resulting from fatigue effects. It should be remembered that the curves in Figs. 142–145 have been plotted for the particular data given in the text. Thermal buckling and temperature stresses, which can have a detrimental effect on the behavior of the panel, were not taken into account.[1]

---

[1] The further development of the theory of non-linear panel flutter is given in the following papers: AMBARTSUMIAN, S. A., BAGDASARIAN, Zh. E. On the stability of orthotropic plates in supersonic gas flow. *Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie*, No. 4 (1961); AMBARTSUMIAN, S. A., BAGDASARIAN, Zh. E. On stability of non-linearly elastic three-layered plates in supersonic gas flow. *Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie*, No. 5 (1961); BOLOTIN, V. V., NOVICHKOV, Iu. N. Buckling and steady flutter of thermally compressed panels in a supersonic flow. *Inzh. zh.* 1, No. 2 (1961); MAKAROV, B. P. On non-linear flutter of plates clamped along the edges, *Proceedings of the conference on the theory of plates and shells*, Kazan', 1961. MAKAROV, B. P. Amplitudes of the steady flutter of clamped panels, *Izv. Vyssh. uchebn. zaved., Mashinostroenie*, No. 5 (1961); NOVICHKOV, Iu. N. Stability and self oscillation of plane and curved panels in a gas flow. *Proceedings of the III Soviet conference on the theory of plates and shells*, Kiev 1962; NOVICHKOV, Iu. N. On the solution of the equations of steady flutter of cylindrical panels. *Inzh. zh.* 2, No. 4 (1962); STEPANOV, R. D. On the question of plate flutter in non-linear formulation. *Izv. Vyssh. uchebn. zaved., Mashinostroenie*, No. 12 (1960); SHVEIKO, Iu. Iu. On the influence of supersonic gas flow on the lower critical force for a cylindrical panel. *Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie* No. 4 (1961). In the following papers are considered also non-stationary problems: BOLOTIN, V. V. Non-stationary flutter of plates and shallow shells in a gas flow. *Izh. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie*, No. 3 (1962); BOLOTIN, V. V. Behavior of heated plates and shells in a gas flow. *Inzh. zh.* 2, No. 3 (1962); NOVICHKOV, Iu. N. On the stability of solution in the problem of panel flutter. *Izv. Akad. Nauk SSSR, OTN, Mekhanika i mashinostroenie*, No. 4 (1962).

CONCLUDING REMARKS. SUGGESTED DIRECTIONS FOR FUTURE RESEARCH

In the preceding chapters we have studied a wide range of non-conservative problems in the theory of elastic stability. Their non-conservative nature is the result of the particular way in which the forces acting on the elastic system depend on the displacements. In order to carry out a stability analysis we need to investigate the eigenvalues of non-self-adjoint boundary value problems; a finite-dimensional analogy of these problems would be a system of equations, the matrices of the coefficients of which in generalized coordinates contain antisymmetric components. This is the difference between such problems and others in which the forces depend explicitly on time, or the self-oscillatory nature of which is brought about by "negative friction".

The investigation of non-conservative problems of elastic stability is closely related with the theory of linear non-self-adjoint differential operators a theory which is still in an early stage of development. Effective methods are needed for finding the complex eigenvalues of the operators which avoid the need to resort to existing methods for self-adjoint boundary-value problems, where solution is facilitated by the appropriate variational principles. We also need methods for studying the dependence of the eigenvalues on one or more parameters, as well as methods which enable us to establish for a non-self-adjoint operator with real coefficients the number of eigenvalues lying in some particular part of the complex plane (for instance, in the right-hand half-plane). In problems of elastic stability we have to deal with operators of the fourth or higher order, but the available theories offer as yet little optimism.

The most effective method of solving non-conservative problems of elastic stability is to reduce the equations to systems with a finite number of degrees of freedom by expressing the required functions in the form of series which satisfy certain conditions, and applying

Galerkin's method. The frequently expressed doubts as to the applicability of Galerkin's method to non-self-adjoint problems are obviously without foundation. In particular, it has been shown in Chapter 4 that the familiar paradox in the problem of membrane flutter is not related directly to the fact that this problem is non-self-adjoint. Nevertheless, it is of interest to investigate the conditions of convergence of Galerkin's method for various classes of non-self-adjoint boundary value problems.

A finite-dimensional analogy of non-self-adjoint boundary-value problems is provided by systems of ordinary differential or algebraic equations which contain circulatory or pseudo-gyroscopic terms (terms with generalized coordinates the matrices of which are anti-symmetric), and have an elementary solution. However, the general principles of these systems have not as yet been fully studied. We can only say that circulatory forces were not reflected in the classical results of Kelvin relating to small oscillations of a finite-dimensional system about its equilibrium position. Some results have recently been formulated in connection with gyroscopic systems where small circulatory increments have the effect of correcting factors.

The greatest theoretical interest is evidently centred in the unique effect of damping in the presence of pseudo-gyroscopic forces, and in particular, in the differences in the results for systems with slight damping which then becomes zero and systems in which damping is absent from the start. We have here a situation which is very similar to that in the theory of stability of laminar flow (the paradoxical difference between the parameters of "viscous" and "non-viscous" instability). In this book we have discovered the dependence of stability in the presence of slight damping, which then vanishes, on the relation between the partial coefficients of damping, and we have investigated the extremum properties of the problem in the absence of damping. These interesting aspects require further study and may perhaps serve as a heuristic basis for obtaining further, more definite, results.

In non-linear problems the most important is the investigation of non-zero solutions in the neighborhood of critical values of parameters corresponding to the intersection of the imaginary axis by the eigenvalues. The small parameter method, we found, is a completely effective means of solving this problem. The nature of the solution (in physical terms, a "soft" or "hard" excitation) depends very much on the properties of the non-linear part of the operator. It is of interest to obtain results analogous to the well-known results relating to the behavior of the eigenfunctions of the non-linear self-adjoint operator in the neighborhood of the point of bifurcation.

So far we have discussed the theoretical aspects of the study of non-conservative problems of the theory of elastic stability. Let us now consider the practical aspects, starting with the problems of stability of equilibrium of elastic systems in the presence of "follower" forces.

It should be noted that the majority of studies devoted to such problems were not prompted by any practical need to apply the results. It is for this reason that in some investigations the non-conservative nature of "follower" forces was not appreciated, which was the source of considerable misunderstanding. Of the forces encountered in engineering practice the forces induced by the pressure or reaction of a jet come nearest to the definition of "follower" forces (provided, of course, the motion of the elastic system compared with the flow of gas is sufficiently slow for the inertia and damping components of the jet to be ignored). It is possible to produce conditions which correspond very closely to the effect of a tangential force or moment on a bar by attaching a nozzle in some appropriate manner. There is no doubt that "follower" forces can have a definite effect on jet and rocket installations, turbines, etc. It is felt that the principal lines along which future research in this field should be directed must aim, not at increasing the number of purely academic problems solved, but at providing an answer to the question of the degree to which "follower" forces can satisfactorily represent actual forces encountered in practice. Here experimental investigations are of prime importance.

In problems of stability of the rectilinear form of a flexible rotor taking into account internal friction, hydrodynamic friction in an oil film and other related factors, the non-conservative nature of the problem is brought about by the presence of a controled motor which maintains a constant speed of rotation. If the displacements of the rotor are small the above factors induce forces which act in a direction normal to the displacement. In the equations of disturbed motion these forces are taken into account by means of terms containing generalized displacements and an antisymmetric matrix of the coefficients. The analogy with other non-conservative problems of elastic stability still holds when the shaft cross-section has unequal principal bending stiffnesses.

The primary concern in problems of this type should be the finding of a more exact method of expressing the forces responsible for energy transfer. This applies not only to hysteresis in the material of a shaft or to friction in clamps, but more to hydrodynamic forces induced in layers of a fluid partially drawn round with the rotor. The investigation of instability phenomena resulting from friction in the oil layer of bearings is probably the most important problem of this type. In addition to experimental investigations, there is a need for a conclusive hydrodynamic theory to be established. We have shown that the various assumptions concerning the nature of the non-linear components of force leads to a wide divergence of the results. Since instability phenomena cannot always be suppressed completely, a further study of the non-linear problem would be of considerable interest.

In this book we have explored the problems of the stability of elastic plates and shells in a flow of gas in more detail. The investigation of the stability of thermally buckled panels is of great practical interest; so also is the investigation of stability when the real laws governing frictional damping are taken into account. It is well known that it is essential to take into account damping in problems with multiple or fairly close partial frequencies. This situation occurs in the case of flat panels elongated in the direction of flow and subjected to compressive forces, and in closed cylindrical shells where the partial frequency spectrum is fairly dense, as well as in a number of other problems.

As a result of the very convenient simplicity of the expression for the aerodynamic forces provided by the "piston theory", the number of publications based on the application of this theory is rapidly increasing. The upper and lower limits of application of the "piston theory" need to be established more reliably. For this reason, further investigations of aeroelastic problems are required in the region of moderate supersonic velocities based on more exact aerodynamic relations. In addition to numerical calculations, attempts should be made to obtain solutions corresponding to the "piston theory" by a limiting process from more exact solutions. By an example of the exact solution of the problem of the stability of a cylindrical shell in a potential flow of gas we have shown in this book that such a limiting process enables us to discover certain additional conditions of application of the "piston theory". Furthermore, there is a need to study aeroelastic problems in the region of high supersonic velocities, taking into account phenomena occurring in the boundary layer.

It has also been shown in the book that the excitation of panel flutter by predominantly geometrical non-linearity is of a predominantly "soft" nature. This means that if a system enters the flutter region, failure of the panel does not follow immediately; it occurs only after a certain number of cycles and is a fatigue type of failure. In contrast, therefore, to the classical problem of wing flutter, where the excitation is "hard", an investigation of the non-linear problem in this case assumes a practical significance. Further investigations are also required in this field. In particular, attention should be given to a study of non-stationary phenomena related to transition through the flutter region. This is a problem which should be investigated in conjunction with the problem of damage accumulation in the structure.

AUTHOR INDEX

Ambartsumian, S. A. 312
Andronov, A. A. 170
Appell, P. E. 170
Ashley, H. 4, 200, 201, 237, 257

Bagdasarian, Zh. E. 312
Bam-Zelikovich, G. M. 237
Barta, J. 89
Bautin, N. N. 268
Beck, M. 4, 88, 128
Bellman, R. E. 43
Bespal'ko, L. A. 141
Biderman, V. L. 120, 128
Biezeno, C. B. 145
Bisplinghoff, R. L. 4, 200, 257
Bocker, G. F. 142
Bolotin, V. V. 4, 5, 9, 55, 64, 71, 82, 89, 104, 142, 157, 162, 165, 169, 200, 201, 202, 214, 226, 229, 247, 258, 266, 275, 280, 289, 291, 312
Budiansky, B. 200
Bunimovich, A. I. 237

Chaevskii, M. I. 141
Chebotarev, N. G. 228
Chernyi, G. G. 237
Chetaev, N. G. 43
Cole, J. E. 191
Courant, R. 31
Cunningham, H. J. 200, 231

Deineko, K. S. 4, 13, 88, 93
Dimentberg, F. M. 4, 141, 149, 157, 160
Duncan, W. J. 199
Dzhanelidze, G. Iu. 4, 87, 88, 104
Dzygadlo, Z. 214

Epishev, L. V. 142, 191
Euler, L. 1

Feodos'ev, V. I. 8, 88, 117, 120, 128
Ferri, A. 237
Floberg, L. 191
Frankl', F. I. 209
Frazer, R. A. 199
Fung, Y. C. 202, 247

Galimov, K. Z. 30, 274
Gavrilov, Iu. V. 5, 202, 280, 291
Goland, M. 200, 235
Gol'denblat, I. I. 28
Gopak, K. N. 89
Grammel, R. 145
Grossman, E. P. 199

Hagg, A. C. 142
Halfman, R. 4, 200, 257
Hayes, W. D. 201, 237
Hedgepeth, J. M. 5, 200, 201, 234, 247
Hilbert, D. 31
Hoff, N. J. 5
Holmes, R. 191
Holt, M. 247
Hori, Y. 190
Huckel, V. 241

Il'yushin, A. A. 201, 237
Ishlinskii, A. Iu. 9

Johns, B. J. 249

Kaliski, S. 214
Kameron, A. 191
Kantorovich, L. V. 263
Kapitsa, P. L. 4, 142
Kappus, R. 33
Karpovich, E. A. 209
Keldysh, M. V. 200
Kestens, J. 191
Kimball, A. L. 140
Kochin, N. E. 36
Kononenko, V. O. 144
Kopeikin, Iu. D. 89
Krivosheeva, S. G. 89
Krylov, V. I. 263
Kushul', M. Ia. 141
Küssner, H. G. 200

Landau, L. D. 70
Lavrent'ev, M. A. 200
Leipholz, H. 89
Leonard, R. W. 200
Leonov, M. Ia. 4, 13, 88, 89, 93, 100, 141
Lewis, J. F. 142
Liapunov, A. M. 43
Librescu, L. 214
Lifshits, E. M. 70
Lighthill, M. J. 201, 237
Likharev, K. K. 120, 128
Livanov, K. I. 247
Luke, Y. L. 200, 235

Makarov, B. P. 5, 202, 247, 280, 291, 312
Makhortykh, Zh. K. 201
Makushin, V. M. 120, 128
Malashenko, S. V. 9
Malinin, N. N. 120, 128
Meiman, N. S. 228
Merkin, D. R. 73, 79
Mikhailova, M. P. 237
Mikhlin, S. G. 50
Mikishev, G. N. 202
Miles, J. W. 201, 231, 234
Morgan, H. G. 241
Morris, J. 4, 87
Mournaghan, F. 28
Movchan, A. A. 5, 201, 244, 247, 256
Mushtari, Kh. M. 274

Naimark, M. A. 50
Narimanov, G. S. 194
Neimark, Iu. I. 171
Nelson, H. C. 200, 231
Newkirk, B. L. 142, 161
Nikolai, B. L. 4, 88, 102
Nikolai, E. L. 2, 4, 86, 128, 141
Novichkov, Iu. N. 312
Novozhilov, V. V. 26, 28, 33

Obmorshev, N. A. 73

Panovko, Ia. G. 151
Papkovich, P. F. 281
Pflüger, A. 4, 8, 46, 88, 89, 93
Poincaré, H. 44
Ponomarev, S. D. 120, 128
Poritsky, H. 142
Pozniak, E. L. 101, 141, 142, 155, 191, 194

Rauscher, M. 199
Rattayya, J. W. 247
Reissner, H. 199
Reut, V. I. 4, 88
Robertson, D. 141
Rocard, Y. 5
Runyan, H. L. 241

Shanley, F. R. 2
Shashkov, I. E. 4, 87, 128
Shen, S. F. 202
Shveiko, Iu. Iu. 5, 202, 247, 280, 291, 312
Smith, D. M. 141, 149
Sokolnikoff, I. S. 36
Sorokin, E. S. 151
Stepanov, R. D. 247, 312
Sternlicht, B. 142

Stewartson, K. 194
Strack, S. L. 247

Taylor, N. D. 142
Temchenko, M. E. 9
Theodorsen, T. 200
Trefftz, E. 30
Troesch, A. 4, 87, 136

Vlasov, V. Z. 105
Vol'mir, A. S. 2, 274
Voss, H. M. 247

Zartarian, C. 201, 237, 257
Zener, C. 177
Ziegler, H. 3, 4, 73, 81, 87, 88, 89, 128
Zorii, L. M. 100

SUBJECT INDEX

Aerodynamic non-linearity 266–268
Aeroelastic problems 199–202
  formulation in general terms 208–214
Aeroelasticity, non-linear problems in 265–274
Automatic control systems, problems in 21

Bar(s)
  compressed by force with fixed line of action, stability of 100–104
  effect of distributed mass of 134–138
  frequencies of natural oscillation of 93
  in compression and torsion
    equations of equilibrium of 119–124
    kinematic relations in 121
    Kirchhoff equations and 122–123, 126
    stability of 124–131
  longitudinal bending of 6
  under tangential force, stability of 90–100
    approximate solution to problem of 95–98
    effect of damping on 95–98
Bar(s)
  under tangential force, stability of
    equation of small oscillations of 90
    influence of mass distribution on 93–94
  with concentrated mass at one end 131–134
Bending
  of a circular arc 89
  plane form of 89
    stability of 104–109
      application of Galerkin's method to 108–109
      numerical results for 111–115
Bifurcation 89
  concepts of 170, 182
  parameters corresponding to 73
  point of 1, 170, 173
Boundary conditions 30–33
Boundary-value problems
  eigenvalues and loading parameter 56
  non-self-adjoint 257
    analysis of eigenvalues of 58
    application of Galerkin's method to 59–62
    eigenvalues of 313